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Abstract

In this paper, we consider the product partition model for the estimation of normal means and variances of a sequence of observations that experiences changes in these parameters at unknown times. The estimates of the parameters by using product partition model are the weighted average of the estimates based in blocks (groups) of observations by the posterior relevance of these blocks which depends on the prior cohesions. We implement the Barry and Hartigan's method to this change point problem and propose an easy-to-implement modification to their method. We use the Yao's prior cohesions and investigate the influence of different prior distributions to the parameter that indexes these cohesions in the product estimates. A comparison between the estimates obtained by using both these methods and those provided by using the Yao's method is done considering different settings for its application. We apply the three methods presented in this paper to stock market data. The results seem to indicate that the method proposed is competitive and also that the prior specifications influence in the product estimates.

Keywords: Change points, product partition model, relevance, Student-t distribution, Yao's cohesions.

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1 Introduction

The product partition model (PPM) proposed by Hartigan (1990) is a good way to model uncertain about a sequence of random quantities, if the prior opinion about it discloses the existence of blocks of observations produced by some judgment of similarities among these observations, as well as independence among the different blocks. In particular, the PPM is an useful tool to analyze change points problems.

The PPM establishes that the random partition produced by the change points has a prior product distribution, and assumes that, given the partition, the parameters in different blocks have independent prior distributions (Barry and Hartigan, 1992). Consequently, the posterior estimates of these parameters (product estimates) are the weighted average of the estimates in each block by the posterior probability that the block appears in the partition, which is called the posterior relevance of the block (see details in Section 2). In general, large number of computations are involved on the product estimates.

The product estimates of the mean of normal random variables with common variance is considered in detail by Barry and Hartigan (1993) using a Gibbs sampling scheme. Barry and Hartigan (1993) compare their estimates with those obtained by Chernoff and Zacks (1964), Yao (1984) and by considering the Schwartz criterion Yao (1988). Barry and Hartigan (1993) conclude that their method provide more accurate estimates if outliers are observed and that Yao's method usualy demand more computational time. Recently, Crowley (1997) provides a new implementation of the Gibbs sampling in order to solve the problem of estimating normal means by using the general PPM. Loschi et al. (1999) extend some results from Crowley (1997) and Barry and Hartigan (1993) by using the PPM to identify multiple change points in the mean and variance of normal data. Loschi et al. (1999) assume the prior cohesions proposed by Yao (1984) and consider Yao's algorithm to compute the product estimates.

This paper discusses the same change point problem considered by Loschi et al. (1999). We also assume the Yao's prior cohesions which depend on the probability that a change occurs at any time. We implement the method established by Barry and Hartigan (1993) and propose an easy-to-implement modification of this method. Similar estimates are obtained as well as the computational time involved in their calculation. We also compare our propose and the Barry and Hartigan's method with Yao's method. Yao's method usually demands less computational time and provides similar results for the product estimates in the application we consider. This contradicts Barry and Hartigan (1993) statements. The ultimate goal is to compare all methods presented here in different contexts in order to percieve the influence in the product estimates of the prior specifications to the parameter involved in the Yao's cohesions.

This paper is organized as follows. Section 2 briefly reviews the PPM introduced by Barry and Hartigan (1992). Section 3 presents inferential solutions to identify change points for random variables which are normally distributed, given the means and variances, according to Loschi et al. (1999). In Section 4 we introduce the computational procedure to calculate the posterior relevances based in a Gibbs sampling approach and review the Yao's algorithm and the method proposed by Barry and Hartigan (1993). In

Section 5, we apply the methods to the two most important Brazilian indexes, *Índice Geral da Bolsa de Valores de São Paulo* (IBOVESPA) and *Índice da Bolsa de Valores de Minas Gerais, Espírito Santo e Brasília* (IBOVMESB), comparing their performance under different prior specifications.

2 Product partition models

In this section, we present a brief revision of the product partition model (PPM), introduced by Barry and Hartigan (1992) to identify multiple change points in a sequence of variables observed at consecutive points in time. A more general definition to PPM can be found in Hartigan (1990).

Let $X_1, \ldots, X_n$ be a observed time series and consider the index set $I = \{1, \ldots, n\}$. Consider a random partition $\rho = \{i_0, i_1, \ldots, i_b\}$ of the set $I \cup \{0\}$, $0 = i_0 < i_1 < \ldots < i_b = n$, and a random variable B to represent the number of blocks in ρ. Consider that each partition divides the sequence $X_1, \ldots, X_n$ into $B = b$ contiguous subsequences, which will be denoted here by $\mathbf{X}_{[i_{r-1}i_j]} = (X_{i_{r-1}+1}, \ldots, X_{i_r})'$, $r = 1, \ldots, b$. Let $c_{[ij]}$ be the prior cohesion associated with the block $[ij] = \{i+1, \ldots, j\}$, $i, j \in I \cup \{0\}$, $j > i$, which represents the degree of similarity among the observations in $\mathbf{X}_{[ij]}$

Hence, we say that the random quantity $(X_1, \ldots, X_n; \rho)$ follows a PPM, denoted by $(X_1, \ldots, X_n; \rho) \sim PPM$, if:

i) the prior distribution of ρ is the following product distribution:

$$P(\rho = \{i_0, \ldots, i_b\}) = \frac{\Pi_{j=1}^{b} c_{[i_{j-1}i_j]}}{\sum_{\mathcal{C}} \Pi_{j=1}^{b} c_{[i_{j-1}i_j]}}, \tag{1}$$

where $\mathcal{C}$ is the set of all possible partitions of the set I into b contiguous blocks with endpoints $i_1, \ldots, i_b$, satisfying the condition $0 = i_0 < i_1 < \ldots < i_b = n$, for all $b \in I$;

ii) conditionally on $\rho = \{i_0, \ldots, i_b\}$, the sequence $X_1, \ldots, X_n$ has the joint density given by:

$$f(X_1, \ldots, X_n | \rho = \{i_0, \ldots, i_b\}) = \Pi_{j=1}^{b} f_{[i_{j-1}i_j]}(\mathbf{X}_{[i_{j-1}i_j]}), \tag{2}$$

where $f_{[ij]}(\mathbf{X}_{[ij]})$ is the density of the random vector, called data factor, $\mathbf{X}_{[ij]} = (X_{i+1}, \ldots, X_j)'$.

Consequently, if $(X_1, \ldots, X_n; \rho) \sim$ PPM, the number of blocks B in ρ has a prior distribution given by:

$$P(B = b) \propto \sum_{\mathcal{C}_1} \Pi_{j=1}^{b} c_{[i_{j-1}i_j]}, \;\; b \in I, \tag{3}$$

where $\mathcal{C}_1$ is the set of all partitions of I in b contiguos blocks with endpoint $i_1, \ldots, i_b$, satisfying the condition $0 = i_0 < i_1 < \ldots < i_b = n$.

As shown in Barry and Hartigan (1992), the posterior distributions of ρ and B have the same form of the prior distribution, where the posterior cohesion for the block $[ij]$ is given by

$$c^*_{[ij]} = c_{[ij]} f_{[ij]}(\mathbf{X}_{[ij]}). \tag{4}$$

In the parametric approach to PPM, a sequence of unknown parameters $\theta_1, \ldots, \theta_n$, such that, conditionally in $\theta_1, \ldots, \theta_n$, the sequence of random variables $X_1, \ldots, X_n$ has conditional marginal densities $f_1(X_1|\theta_1), \ldots, f_n(X_n|\theta_n)$, respectively, is considered. The prior distribution of $\theta_1, \ldots, \theta_n$ is constructed as follows. Given a partition $\rho = \{i_0, \ldots, i_b\}$, $b \in I$, we have that $\theta_i = \theta_{[i_{r-1} i_r]}$ for every $i_{r-1} < i \leq i_r$, $r = 1, \ldots, b$, and that $\theta_{[i_0 i_1]}, \ldots, \theta_{[i_{b-1} i_b]}$ are independent, with $\theta_{[ij]}$ having (block) prior density $\pi_{[ij]}(\theta)$, $\theta \in \Theta_{[ij]}$, where $\Theta_{[ij]}$ is the parameter space corresponding to the common parameter, say, $\theta_{[ij]} = \theta_{i+1} = \ldots = \theta_j$, which indexes the conditional density of $\mathbf{X}_{[ij]} = (X_{i+1}, \ldots, X_j)'$. In this case, we consider that two observations X_i and X_j, $i \neq j$, are in the same block, if they are identically distributed. Thus, in this approach to PPM, the predictive distribution $f_{[ij]}(X_{[ij]})$, which appeared in (2), can be obtained as follows:

$$f_{[ij]}(\mathbf{X}_{[ij]}) = \int_{\Theta_{[ij]}} f_{[ij]}(\mathbf{X}_{[ij]}|\theta)\pi_{[ij]}(\theta)d\theta, \tag{5}$$

The goal is to obtain the marginal posterior distributions of the parameters ρ, B, and θ_k, $k = 1, \ldots, n$. The posterior distributions of ρ and B are obtained as described before and considering the joint density given in (5). Barry and Hartigan (1992) have shown that the posterior distributions of θ_k is given by:

$$\pi(\theta_k|X_1, \ldots, X_n) = \sum_{i=0}^{k-1}\sum_{j=k}^{n} r^*_{[ij]}\pi_{[ij]}(\theta_k|\mathbf{X}_{[ij]}), \; k = 1, \ldots, n, \tag{6}$$

and the posterior expectation (or product estimate) of θ_k is given by:

$$E(\theta_k|X_1, \ldots, X_n) = \sum_{i=0}^{k-1}\sum_{j=k}^{n} r^*_{[ij]}E(\theta_k|\mathbf{X}_{[ij]}), \; k = 1, \ldots, n, \tag{7}$$

where $r^*_{[ij]}$ denotes the posterior relevance for the block $[ij]$, that is:

$$r^*_{[ij]} = P([ij] \in \rho|X_1, \ldots, X_n),$$

which, in the situation introduced by Barry and Hartigan (1993) and briefly decribed in this section, become:

$$r^*_{[ij]} = \frac{\lambda_{[0i]}c^*_{[ij]}\lambda_{[jn]}}{\lambda_{[0n]}}, \tag{8}$$

with $\lambda_{[ij]} = \sum \Pi_{k=1}^{b} c^*_{[i_{k-1} i_k]}$, where the summation is over all partitions of $\{i+1, \ldots, j\}$ in b blocks with endpoints $i_0, i_1, \ldots, i_b$ satisfying the condition $i = i_0 < i_1 < \ldots < i_b = j$.

In algorithmic language, the PPM could be stated as shown in Figure 1.

Figure 1 goes around here

3 Posterior estimates for the normal means and variances

To specify the PPM for the normal case, Loschi et al. (1999) assume that there is a sequence of unknown parameters $\theta_1 = (\mu_1, \sigma_1^2), \ldots, \theta_n = (\mu_n, \sigma_n^2)$, such that $X_k|\mu_k, \sigma_k^2 \sim \mathcal{N}(\mu_k, \sigma_k^2)$, $k = 1, \ldots, n$, and that are independent.

It is also assumed that each common parameter $\theta_{[ij]} = (\mu_{[ij]}, \sigma^2_{[ij]})$, related to the block $[ij]$, has the conjugate normal-inverted-gamma prior distribution denoted by:

$$(\mu_{[ij]}, \sigma^2_{[ij]}) \sim \mathcal{NIG}(m_{[ij]}, v_{[ij]}; a_{[ij]}/2, d_{[ij]}/2),$$

that is,

$$\mu_{[ij]}|\sigma^2_{[ij]} \sim \mathcal{N}(m_{[ij]}, v_{[ij]}\sigma^2_{[ij]}) \quad \text{and} \quad \sigma^2_{[ij]} \sim \mathcal{IG}(a_{[ij]}/2, d_{[ij]}/2), \tag{9}$$

where $\mathcal{IG}(a, d)$ denotes the inverted-gamma distribution with parameters a and d, $m_{[ij]} \in \mathcal{R}$, and $a_{[ij]}$, $d_{[ij]}$ and $v_{[ij]}$ are positive values. Hence, the conditional distribution of $\theta_{[ij]} = (\mu_{[ij]}, \sigma^2_{[ij]})$, given the observations in $\mathbf{X}_{[ij]}$, is the normal-inverted-gamma distribution given by:

$$(\mu_{[ij]}, \sigma^2_{[ij]})|\mathbf{X}_{[ij]} \sim \mathcal{NIG}(m^*_{[ij]}, v^*_{[ij]}; a^*_{[ij]}/2, d^*_{[ij]}/2), \tag{10}$$

where

$$\left.\begin{array}{rcl} m^*_{[ij]} & = & \frac{(j-i)v_{[ij]}\bar{X}_{[ij]}}{(j-i)v_{[ij]}+1} + \frac{m_{[ij]}}{(j-i)v_{[ij]}+1}, \\ v^*_{[ij]} & = & \frac{v_{[ij]}}{(j-i)v_{[ij]}+1}, \\ d^*_{[ij]} & = & d_{[ij]} + j - i, \\ a^*_{[ij]} & = & a_{[ij]} + q_{[ij]}(\mathbf{X}_{[ij]}), \end{array}\right\} \tag{11}$$

with

$$\begin{array}{rcl} \bar{X}_{[ij]} & = & \dfrac{1}{j-i}\displaystyle\sum_{r=i+1}^{j} X_r, \\ q_{[ij]}(\mathbf{X}_{[ij]}) & = & \displaystyle\sum_{r=i+1}^{j} (X_r - \bar{X}_{[ij]})^2 + \dfrac{(j-i)(\bar{X}_{[ij]} - m_{[ij]})^2}{(j-i)v_{[ij]} + 1}. \end{array}$$

Consequently, it follows from (10) that, given $X_{[ij]}$, the conditional marginal densities of $\mu_{[ij]}$ and $\sigma^2_{[ij]}$ are, respectively:

$$\mu_{[ij]}|\mathbf{X}_{[ij]} \sim t(m^*_{[ij]}, v^*_{[ij]}, a^*_{[ij]}, d^*_{[ij]}) \text{ and } \sigma^2_{[ij]}|\mathbf{X}_{[ij]} \sim \mathcal{IG}(a^*_{[ij]}/2, d^*_{[ij]}/2), \tag{12}$$

for which it is observed that

$$E(\mu_{[ij]}|X_{[ij]}) = m^*_{[ij]} \ \ (\text{if } d^*_{[ij]} > 1) \tag{13}$$

and

$$E(\sigma^2_{[ij]}|X_{[ij]}) = \frac{a^*_{[ij]}}{d^*_{[ij]} - 2} \ \ (\text{if } d^*_{[ij]} > 2). \tag{14}$$

The interested reader may find details in O'Hagan (1994).

From (7), (13) and (14), it follows that the product estimates for the parameters μ_k and σ_k^2, $k = 1, \ldots, n$, are given by:

$$E(\mu_k|X_1,\ldots,X_n) = \sum_{i=0}^{k-1}\sum_{j=k}^{n} r^*_{[ij]} m^*_{[ij]} \quad (\text{if } d^*_{[ij]} > 1) \tag{15}$$

and

$$E(\sigma_k^2|X_1,\ldots,X_n) = \sum_{i=0}^{k-1}\sum_{j=k}^{n} r^*_{[ij]} \frac{a^*_{[ij]}}{d^*_{[ij]} - 2} \quad (\text{ if} d^*_{[ij]} > 2), \tag{16}$$

respectively, where $m^*_{[ij]}$, $a^*_{[ij]}$ and $d^*_{[ij]}$ are defined as in (11).

Let $\mathbf{1}_n$ be the $n \times 1$ vector of ones and $\mathbf{I}_n$ the $n \times n$ identity matrix. The posterior relevances $r^*_{[ij]}$ can be obtained from (8) and (4) where the random vector $\mathbf{X}_{[ij]}$ follows a $(j-i)$-dimensional Student-t distribution denoted by $\mathbf{X}_{[ij]} \sim t_{j-i}(\mathbf{m}_{[ij]}, \mathbf{V}_{[ij]}; a_{[ij]}, d_{[ij]})$ with density function given by

$$f(\mathbf{X}_{[ij]}) = c(d_{[ij]}, j-i) a_{[ij]}^{d_{[ij]}/2} |\mathbf{V}_{[ij]}|^{-1/2} \{a_{[ij]} + (\mathbf{X}_{[ij]} - \mathbf{m}_{[ij]})' \mathbf{V}_{[ij]}^{-1} (\mathbf{X}_{[ij]} - \mathbf{m}_{[ij]})\}^{-(d_{[ij]}+j-i)/2}, \tag{17}$$

where $c(d,k) = \Gamma[\frac{d+k}{2}]\{\Gamma[\frac{d}{2}]\,\pi^{\frac{k}{2}}\}^{-1}$ and $\mathbf{m}_{[ij]} = m_{[ij]}\mathbf{1}_{j-i}$ and $\mathbf{V}_{[ij]} = \mathbf{I}_{j-i} + v_{[ij]}\mathbf{1}_{j-i}\mathbf{1}'_{j-i}$.

The algorithm of this normal case is depicted in Figure 2.

Figure 2 goes aroud here

4 Computational procedures

Notice from (15) and (16) that high computational efforts are demanded to calculate the product estimates. To simplify these calculations some procedures were proposed in the literature. In this section, we review the computational approach developed by Yao (1984) and introduce a Gibbs sampling scheme to compute the posterior relevances (and, consequently, the product estimates). We also describe the Barry and Hartigan (1993) method to calculate the product estimates of the means and variances by using PPM.

We will assume the prior cohesions suggested by Yao (1984) and defined below, since in the PPM shown in this paper the cohesions can be interpreted as transition probabilities in the Markov chain defined by the endpoints of the blocks in ρ.

Let p, $0 \le p \le 1$, be the probability that a change occurs at any instant in the sequence. Therefore, the prior cohesion for block $[ij]$ is given by:

$$c_{[ij]} = \begin{cases} p(1-p)^{j-i-1}, & \text{if } j < n, \\ (1-p)^{j-i-1}, & \text{if } j = n, \end{cases} \tag{18}$$

for all $i,\ j \in I$, $i < j$, which corresponds to the probability that a new change takes place after $j-i$ instants, given that a change has taken place at instant i.

Consequently, for the normal case presented in Section 3, the posterior cohesion of the block $[ij]$ become:

$$c^*_{[ij]} = \begin{cases} \dfrac{p(1-p)^{j-i-1} c(d_{[ij]}, j-i) a_{[ij]}^{d_{[ij]}/2}}{(1+(j-i)v_{[ij]})^{1/2}\{a_{[ij]} + q_{[ij]}(\mathbf{X}_{[ij]})\}^{(d_{[ij]}+j-i)/2}}, & \text{if } j < n \\ \dfrac{(1-p)^{j-i-1} c(d_{[ij]}, j-i) a_{[ij]}^{d_{[ij]}/2}}{(1+(j-i)v_{[ij]})^{1/2}\{a_{[ij]} + q_{[ij]}(\mathbf{X}_{[ij]})\}^{(d_{[ij]}+j-i)/2}}, & \text{if } j = n, \end{cases} \tag{19}$$

4.1 Yao's Algorithm

Let $\lambda_{[ij]}$ be the summation presented in (8) where $c^*_{[ij]}$ is the posterior cohesions given in (19). Hence, the exact posterior relevances given in (8), can be obtained by using the following recursive algorithm:

$$\left.\begin{array}{rcl} \lambda_{[00]} & = & 1, \\ \lambda_{[01]} & = & c^*_{[01]}, \\ \lambda_{[0j]} & = & c^*_{[0j]} + \sum_{t=1}^{j-1} \lambda_{[0t]} c^*_{[tj]}, \ \forall j = 2, \ldots, n, \\ \lambda_{[(n-1)n]} & = & c^*_{[(n-1)n]}, \\ \lambda_{[in]} & = & c^*_{[in]} + \sum_{t=i+1}^{n-1} \lambda_{[tn]} c^*_{[it]}, \ \forall i = 1, \ldots, n-2, \\ \lambda_{[nn]} & = & 1. \end{array}\right\} \quad (20)$$

Notice that in spite of the simplifications introduced by Yao (1984), great computational efforts are still demanded in PPM. In Section 4.2, we propose a procedure to simplify the computation of the posterior relevances and, consequently, the product estimates using the tranformation suggested by Barry and Hartigan (1993). We also describe the Barry and Hartigan's method to obtain the product estimates of normal means and variances.

4.2 Gibbs Sampling Schemes to Compute Posterior Relevances

In order to estimate the posterior relevances of each block $[ij]$ and the posterior distibution of ρ using a Gibbs sampling scheme, we consider the transformation suggested by Barry and Hartigan (1993). Let assume the auxiliary random quantity U_i which reflects whether or not a change point occurres at the time i, that is:

$$U_i = \begin{cases} 1, & \text{if } \theta_i = \theta_{i+1}, \\ 0, & \text{if } \theta_i \neq \theta_{i+1}, \end{cases}$$

$i = 1, \ldots, n-1$. Notice that the random partition ρ is perfectly identified by considering vectors $\mathbf{U} = (U_1, \ldots, U_{n-1})$ of these random quantities. Consequently, the posterior relevance of the block $[ij]$, $i < j$, can be estimated from the number of vectors where is observed $U_i = 0$, $U_{i+1} = \ldots, U_{j-1} = 1$ and $U_j = 0$.

Each vector $(U_1^k, \ldots, U_{n-1}^k)$, $k \geq 1$, is generated by using the Gibbs sampling as follows. Starting with an initial values $(U_1^0, \ldots, U_{n-1}^0)$ of $\mathbf{U}$, at step k, the r-th element U_r^k is generated from the conditional distribution:

$$U_r | U_1^k, \ldots, U_{r-1}^k, U_{r+1}^{k-1}, \ldots, U_{n-1}^{k-1}; X_1, \ldots, X_n,$$

$r = 1, \ldots, n-1$. In order to generate the samples above, it is sufficient to consider the following ratio:

$$R_r = \frac{P(U_r = 1 | A_k; X_1, \ldots, X_n)}{P(U_r = 0 | A_r^k; X_1, \ldots, X_n)},$$

$r = 1, \ldots, n-1$, where $A_r^k = \{U_1^k = u_1, \ldots, U_{r-1}^k = u_{r-1}, U_{r+1}^{k-1} = u_{r+1}, \ldots, U_{n-1}^{k-1} = u_{n-1}\}$.

Consequently, the criterion of choosing the values U_i^k, $i = 1, \ldots, n-1$ becomes:

$$U_r^k = \begin{cases} 1, & \text{if } R_r \geq \frac{1-u}{u} \\ 0, & \text{otherwise,} \end{cases}$$

where $r = 1, \ldots, n-1$ and $u \sim \mathcal{U}(0,1)$.

Consider the prior cohesions given in (18) and assume that p has the prior distribution $\pi(p)$. Let $\mathcal{C}$ be the set of all partitions of the set I into b contiguous blocks with endpoint $i_0, \ldots, i_b$ satisfying the condition $0 = i_0 < i_1 < \ldots < i_b = n$, $b \in I$ and consider $\mathcal{C}_1 \subset \mathcal{C}$ the subset of all partitions that contain the block $[ij] = \{i+1, \ldots, j\}$. Thus, each value U_r^k, $k \geq 1$, $r = 1, \ldots, n-1$, can be generated by using

$$R_r = \frac{f_{[xy]}(X_{[xy]}) \int_0^1 p^{b-2}(1-p)^{n-b+1} d\pi(p)}{f_{[xr]}(X_{[xr]}) f_{[ry]}(X_{[ry]}) \int_0^1 p^{b-1}(1-p)^{n-b} d\pi(p)}, \tag{21}$$

where:

$$x = \begin{cases} \max\{i, \textbf{ s.t.: } 0 < i < r, U_i^k = 0\}, & \text{if } U_i^k = 0, \text{ for some } i \in \{1, \ldots, r-1\} \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

and

$$y = \begin{cases} \min\{i, \textbf{ s.t.: } r < i < n, U_i^{k-1} = 0\}, & \text{if } U_i^{k-1} = 0, \text{ for some } i \in \{r+1, \ldots, n-1\} \\ n, & \text{otherwise.} \end{cases} \tag{23}$$

According to Loschi et al. (2001a), if p has the beta prior distribution with $\alpha > 1$ and $\beta > 1$ parameters, denoted by $p \sim \mathcal{B}(\alpha, \beta)$, the value R_r given in (21) become:

$$R_r = \frac{f_{[xy]}(X_{[xy]}) \Gamma(n+\beta-b+1) \Gamma(b+\alpha-2)}{f_{[xr]}(X_{[xr]}) f_{[ry]}(X_{[ry]}) \Gamma(b+\alpha-1) \Gamma(n+\beta-b)}, \; b = 1, \ldots, n, \tag{24}$$

where x and y is obtained as in (22) and (23), respectively.

If no prior distribution is considered to p, the value R_r used in the generation of U_r^k can be obtained by using the following ratio:

$$R_r = \frac{c^*_{[xy]}}{c^*_{[xr]} c^*_{[ry]}}, \tag{25}$$

where x and y are given in (22) and (23).

Notice that, if the normal case described in Section 3 is considered, the joint density $f_{[ij]}(X_{[ij]})$ is the Student-t distribution given in (17) and the posterior cohesions $c^*_{[ij]}$ are given in (19).

In both situations, the product estimates of μ_k and σ_k^2, $k = 1, \ldots, n$, can be obtained by estimating the posterior relevances as proposed in this section and using the formulas presented in (15) and (16), respectively.

Figure 3 shows the algorithm described above, in pseudo-language.

Figure 3 goes around here

4.3 Barry and Hartigan's Method

Barry and Hartigan (1993) consider the Gibbs sampling described in the previous section and they obtain the product estimates of μ_k and σ_k^2, $k = 1, \dots, n$, as follows. For each partition $(U_1^k, \dots, U_{n-1}^k)$, $k \geq 1$, the estimates of μ_k and σ_k^2, $k = 1, \dots, n$, are computed by using (13) and (14), respectively. The product estimates of μ_k and σ_k^2, $k = 1, \dots, n$, are approximated by the average of the values obtained.

The Barry and Hartigan's algorithm is presented in Figure 4.

Figure 4 goes around here

4.4 Remarks on the algorithms

There is no essencial difference between our method and Barry and Hartigan's method. However, the algorithm we propose is imediatelly linked with the theorical statements because, like Yao's algorithm, we propose a method to calculate the posterior relevances. From Figures 3 and 4, we also notice that our method is easy to implement and its efficiency is similar to Barry and Hartigan's method (see the comparision in Section 5).

5 Implementation and analysis

The aim of this section is to compare the different algorithms described in Section 4. The method we propose and the Barry and Hartigan's method are implemented considering three different prior specifications for the parameter p. Firstly, we assume that p is a fixed value arbitrarely choose. After that, we consider that p has a beta prior distribution. Finally, we assume a non-informative prior distribution for p.

As it is well known, Yao's method does not consider prior distributions for p. That is, to use Yao's method, we ought to be sure about the value of p. Then, to fairly compare the methods, we consider different values of p. Beta distributions whose modal values are close to the selected p are also assumed.

Our applications focus on the identification of multiple change points in the mean (expected return) and variance (volatility) in stock market return series. We consider the two most important Brazilian indexes, *Índice Geral da Bolsa de Valores de São Paulo* (IBOVESPA) and *Índice da Bolsa de Valores de Minas Gerais, Espírito Santo e Brasília* (IBOVMESB), within the period from January, 1991 to August, 1999.

As usual in finance, a return series is defined by using the transformation $R_t = (P_t - P_{t-1})/P_{t-1}$, where P_t is the closing price in the month t. IBOVESPA and IBOVMESB return series are plotted all together in Figure 5.

Figure 5 goes around here

From Figure 5, it is noticeable that IBOVESPA and IBOVMESB series present a similar behavior, suggesting the existence of some changes in the mean and variance of the returns in both series. Despite the similarities, IBOVMESB series presents a considerably different return in January, 1992. We are also interested in identifing which method would work better in such a situation.

We suppose that returns are conditionally independent and distributed according to the normal distribution $\mathcal{N}(\mu_{[ij]}, \sigma^2_{[ij]})$, and adopt the natural conjugate prior distribution for the parameters $\mu_{[ij]}$ and $\sigma^2_{[ij]}$ which, in this case, is a normal-inverted-gamma distribution.

In accordance to Loschi et al. (2001b) specifications, the following normal-inverted-gamma prior distribution is adopted to describe the uncertainty on the parameter $(\mu_{[ij]}, \sigma^2_{[ij]})$ for both indexes:

$$\mu_{[ij]}|\sigma^2_{[ij]} \sim \mathcal{N}(0, \sigma^2_{[ij]}), \text{ and } \sigma^2_{[ij]} \sim \mathcal{IG}\left(\frac{0.01}{2}, \frac{4}{2}\right).$$

A small number of change points is expected for both indexes which implies that p should assume lower values with higher probability. Despite this, to observe the influence of the prior specifications of p, we will consider $p = 0.1, 0.5, 0.9$ and also the following beta prior distributions of p: $p \sim \mathcal{B}(5, 50)$, $p \sim \mathcal{B}(50, 50)$ and $p \sim \mathcal{B}(50, 5)$, which are plotted in Figure 6.

Figure 6 goes around here

In the Gibbs sampling scheme, we generate 10,000 samples of $\mathbf{U}$ vector with dimension 103, starting from a vector of zeros, and discharged the inicial 4,000 interations. A lag of 10 is selected in order to avoid correlation among vectors, which means that we worked with a net sample size of 600. See details about Gibbs sampling practical implementations in Gamerman (1997). In the Appendix, we discuss the effect in the product estimates of sample size, the burn-in period and the lag to be chosen.

5.1 Results analysis

The algorithms presented in Section 4 was coded in $C++$ and they are available from the authors upon request. All tests were performed in a PC like computer 166 MHz and 32 MB RAM.

5.1.1 IBOVESPA case

In this secion, we present the posterior estimates of the mean return and the volatility of IBOVESPA series obtained by using all methods described in Section 4. Firstly, we present these estimates considering p fixed. The computational time was close to 14 seconds for our method and Barry and Hartigan's method, and less then 1 second, for Yao's method.

Notice from Figures 7, 8 and 9 that the posterior estimates obtained by using the three methods are similar for each value of p. For $p = 0.5$ and $p = 0.9$, the PPM identified almost every point of time as a change point which does not correspond to the expectation for the Brazilian stock market. In general, this fact is observed for high values of p.

Figures 7, 8 and 9 go around here

Figures 10, 11 and 12 show the product estimates obtained by the method we propose and Barry and Hartigan's method, if the beta prior distributions of p, presented in Figure 6, are considered. The computational time was 25 seconds for both methods. The posterior estimates are also compared with those obtained by Yao's method, considering a value of p close to the modal value of the each beta distribution elicited. Notice that the three methods produce similar posterior estimates in all cases.

Figures 10, 11 and 12 go around here

As the use of non-informative prior distributions are common in Bayesian statistics, it is important to discuss the performance of Barry and Hartigan's method and our method if p has an uniforme prior distribution in the $(0, 1)$ interval, denoted by $p \sim \mathcal{U}(0, 1)$. In Figures 13, 14 and 15 the product estimates obtained by using both methods are compared with those obtained by considering Yao's method assuming $p = 0.1, 0.5, 0.9$, respectivelly.

Figures 13, 14 and 15 go around here

As in the other scenarios considered before, we notice that the product estimates obtained by using Barry and Hartigan's method and our method are coincident for each p. However, in this case, the posterior estimates indicate the existence of a hight number of change points, as we observe in those cases in which the prior specifications for p consider more mass to high values.

It is also important to notice that, if a non-informative prior distribution is assumed, the posterior estimates obtained by the three methods tend to be equal, if we consider values of p close to one in the implementation of Yao's method.

Figures 16 and 17 show the product estimates of the mean and variance obtained, respectivelly, by our method and Barry and Hartigan's method. It is assumed in both cases $p = 0.1$, $p \sim \mathcal{B}(5, 50)$ and $p \sim \mathcal{U}(0, 1)$. We notice that the adoption of this beta prior distribution produces the same product estimates then those obtained by considering a fixed value to p independently of the method we choose. Others beta prior distributions which concentrate the most of the mass in lower values of p were considered, producing very similar estimates (see an example in Figure 18). Similar behaviors not shown are observed for the other scenes used before in this paper.

Figures 16, 17 and 18 go around here

5.1.2 IBOVMESB case

IBOVESPA and IBOVMESB series present similar behavior but in IBOVMESB series we observe a considerably different return in January, 1992 (see Figure 5). Barry and Hartigan (1993) conclude that, considering a completely Bayesian approach, in which hiperprior distributions are assumed, their method works better the Yao's method, in the identification of atipical observations.

Figures 19, 20 and 21 present the product estimates considering $p = 0.1$, $p \sim \mathcal{B}(5, 50)$ and $p \sim \mathcal{U}(0, 1)$, respectively, for our method and Barry and Hartigan's method. For Yao's method, we consider $p = 0.1$ in all circunstances. Notice that, if no prior distribution is assumed, our method and Barry and Hartigan's method works better then Yao's method in estimating the mean and variance of the atipical observation occurred in January, 1992. Yao's method also provide higher estimates to the variance during 1991. From January, 1992 on, the same product estimates were obtained for all methods. The same analysis can be done for beta prior case presented in Figure 20. Similarly to IBOVESPA series, if an uniform prior distribution is considered (see Figure 21) or if the prior specifications that consider high mass to high values of p are considered, almost all points are identified as a change point and the atipical observation is not identified.

Figures 19, 20 and 21 go around here

5.2 Data analysis

According to Loschi et al. (2001b), a small number of change points is expected for the Brazilian stock markets. Then, to analyse the data, we consider prior specifications for p which considers most mass in small values.

We notice from Figures 7 and 19, for example, that change points observed in IBOVESPA and IBOVMESB series typically occur at the same time and that the changes are in the same direction. However, some differences in the behavior of these series are observed. The two changes observed in IBOVMESB series, in August and October, 1991, do not occur in IBOVESPA series. These change points could be related to the sale of USIMINAS, a very important state steel company, located in Minas Gerais state. In October, 1991, USIMINAS was sold for a private group. The beginning of the crisis in the Fernando Collor's government in March, 1992, which culminate with his impeachment, in December of the same year, could be the events that produced the change points in IBOVMESB series, around these two months. Unlikewise the initial expectations, these important historical facts do not seem to produce changes in the behavior of IBOVESPA series.

In July, 1999, Russia's crisis could have produced the change in the IBOVMESB series. However, we do not observe changes in the IBOVESPA series within that period. This different behavior could be explained by the policy adopted by Brazilian government during Asias's crisis, in August, 1997, and because IBOVESPA is the main indicator of Brazilian economy, incorporating the benefits of the government policies more immediately.

A new currency, the Real, was introduced in July, 1994. The Real period has presented lower expected returns and volatilities than the previous period. Mexico, and Asia's crises might be responsible for the market warm-up observed, in January, 1995 and August, 1997, respectively. We notice that the periods when higher volatility was observed during the Real period have been smaller than in the preceding period. Some political actions of the Minas Gerais State Governor, in January, 1999, could be associated with the decrease of the expected returns and volatilities of both indexes, from this period on.

6 Final Comments

We proposed an easy-to-implement method to compute posterior relevances and, consequently, to calculate the product estimates, considering different prior specifications to the parameter that indexes Yao's cohesions. We applied this method to identify change points in normal means and variances. We also implemented

Barry and Hartigan's method to the same change point problem. Both methods were compared with Yao's method within different settings. The effect of different prior specifications in the product estimates were studied.

We conclude that our method and Barry and Hartigan's method produce the same posterior estimates and take the same computational time. Contradicting the statements of Barry and Hartigan (1993), we obtained that, in general, Yao's method produces comparable estimates taking less computational time for every scenario we consider. However, our conclusion about the efficiency of Yao's method in the identification of atipical observations agree with the statements of Barry and Hartigan (1993). Our method and Barry and Hartigan's method are more sensitive to the presence of outliers.

Other advantage of our method and Barry and Hartigan's method is the possibility of eliciting prior distributions to p, that can give more flexibility in the use of the PPM. For exampla, we observe that the product estimates tend to be close if beta distributions having the most of their mass concentrated in small values are considered.

The use of non-informative prior distributions to p as well as the use of prior specifications to p that consider high mass to high values of p identify almost every point as a change point, which for the Brazilian stock market is not an appropriated choice. We also notice that by using these prior specifications the methods do not identify the atipical observations.

Finally, we observe that the PPM works well in the identification of change points in the Brazilian stock market if an appropriated prior specification is done.

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Appendix

For the discussion about the Gibbs sampling scheme implemented here, we apply our method to IBOVESPA series and assume that $p \sim \mathcal{B}(5, 50)$. Similar results (not shown) were obtained for Barry and Hartigan's method and for the others prior specifications adopted in this paper.

We consider the scenarios presented in Table 1.

Table 1: Scenarios Generated

sample #	total size	burn-in	lag	net size
1	1,000	0	1	1,000
2	1,000	100	1	900
3	1,000	0	10	100
4	600	0	1	600
5	1,000	400	1	600
6	10,000	4,000	10	600

Figure 22 shows the product estimates obtained by generating 1,000 samples of **U** and different values for the burn-in period and the lag. Figure 23 presents the product estimates for a net sample size of 600, generated by using different sizes, burn-in periods, and lags. We observe that there is not influence of these specifications in product estimates. However, the computational time decreases dramaticly, if we take small sample size, making our method and Barry and Hartigan's method as efficient as Yao's method. For sample # 4, for example, the computational time was 2 seconds. Samples # 1, # 2, # 3, and # 5 take about 3 seconds. Sample # 6 demanded 25 seconds.

Figures 22 and 23 go around here

A detailed discussion about Markov chain Monte Carlo (MCMC) technics can be found, *e.g.*, in Robert and Casella (1999), Gamerman (1997) and Gilks et al. (1996).

Figure Captions

Figure 1: Main Algorithm

Figure 2: Main Algorithm (μ and σ $\mathcal{N}$ormal Case)

Figure 3: Proposed Algorithm (μ and σ $\mathcal{N}$ormal Case)

Figure 4: Barry and Hartigan Method (μ and σ $\mathcal{N}$ormal Case)

Figure 5: IBOVESPA and IBOVMESB Return Series

Figure 6: Beta distributions

Figure 7: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p = 0.1$

Figure 8: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p = 0.5$

Figure 9: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p = 0.9$

Figure 10: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{B}(5, 50)$

Figure 11: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{B}(50, 50)$

Figure 12: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{B}(50, 5)$

Figure 13: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{U}(0, 1)$ and $p = 0.1$

Figure 14: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{U}(0, 1)$ and $p = 0.5$

Figure 15: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{U}(0, 1)$ and $p = 0.9$

Figure 16: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - Proposed Method

Figure 17: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - Barry and Hartigan's Method

Figure 18: Product Estimates Considering Different Beta Prior Distributions - IBOVESPA (Proposed Method)

Figure 19: Product Estimates to the Expected Returns and Volatilities of IBOVMESB- $p = 0.1$

Figure 20: Product Estimates to the Expected Returns and Volatilities of IBOVMESB - $p \sim \mathcal{B}(5, 50)$ and $p = 0.1$

Figure 21: Product Estimates to the Expected Returns and Volatilities of IBOVESPA - $p \sim \mathcal{U}(0,1)$ and $p = 0.1$

Figure 22: Product Estimates using different sizes to the net sample

Figure 23: Product Estimates using net sample size igual to 600

Figures

algorithm
  read $X_1, \ldots, X_n$
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
(1) $f_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \int_{\Theta_{[ij]}} f_{[ij]}(\mathbf{X}_{[ij]}|\theta)\pi_{[ij]}(\theta)d\theta$
(2) $c^*_{[ij]} \leftarrow c_{[ij]} f_{[ij]}(\mathbf{X}_{[ij]})$
  end for
(3) *compute* $\begin{cases} \lambda_{[0j]}, \forall j = 0, \ldots, n; \\ \lambda_{[in]}, \forall i = 1, \ldots, n-1; \end{cases}$
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
(4) $r^*_{[ij]} \leftarrow \frac{\lambda_{[0i]} c^*_{[ij]} \lambda_{[jn]}}{\lambda_{[0n]}}$
  end for
  for $k = 1$ **to** n **do**
(5) $E(\theta_k|X_1, \ldots, X_n) \leftarrow \sum_{i=0}^{k-1} \sum_{j=k}^{n} r^*_{[ij]} E(\theta_k|\mathbf{X}_{[ij]})$
  end for
  write $E(\theta_1), \ldots, E(\theta_n)$
end algorithm

Figure 1:

algorithm
  read $X_1, \ldots, X_n$
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $f_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \int_{\Theta_{[ij]}} f_{[ij]}(\mathbf{X}_{[ij]}|\theta)\pi_{[ij]}(\theta)d\theta$
    $c^*_{[ij]} \leftarrow c_{[ij]} f_{[ij]}(\mathbf{X}_{[ij]})$
  end for
  compute $\begin{cases} \lambda_{[0j]}, \forall j = 0, \ldots, n; \\ \lambda_{[in]}, \forall i = 1, \ldots, n-1; \end{cases}$
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $r^*_{[ij]} \leftarrow \frac{\lambda_{[0i]} c^*_{[ij]} \lambda_{[jn]}}{\lambda_{[0n]}}$
    $\bar{X}_{[ij]} \leftarrow \frac{1}{j-i} \sum_{r=i+1}^{j} X_r$
    $m^*_{[ij]} \leftarrow \frac{(j-i) v_{[ij]} \bar{X}_{[ij]}}{(j-i) v_{[ij]} + 1} + \frac{m_{[ij]}}{(j-i) v_{[ij]} + 1}$
    $v^*_{[ij]} \leftarrow \frac{v_{[ij]}}{(j-i) v_{[ij]} + 1}$
    $d^*_{[ij]} \leftarrow d_{[ij]} + j - i$
    $q_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \sum_{r=i+1}^{j} (X_r - \bar{X}_{[ij]})^2 + \frac{(j-i)(\bar{X}_{[ij]} - m_{[ij]})^2}{(j-i) v_{[ij]} + 1}$
    $a^*_{[ij]} \leftarrow a_{[ij]} + q_{[ij]}(\mathbf{X}_{[ij]})$
  end for
  for $k = 1$ **to** n **do**
    $E(\mu_k | X_1, \ldots, X_n) \leftarrow \sum_{i=0}^{k-1} \sum_{j=k}^{n} r^*_{[ij]} m^*_{[ij]}$
    $E(\sigma^2_k | X_1, \ldots, X_n) \leftarrow \sum_{i=0}^{k-1} \sum_{j=k}^{n} r^*_{[ij]} \frac{a^*_{[ij]}}{d^*_{[ij]} - 2}$
  end for
  write $E(\mu_1), E(\sigma^2_1), \ldots, E(\mu_n), E(\sigma^2_n)$
end algorithm

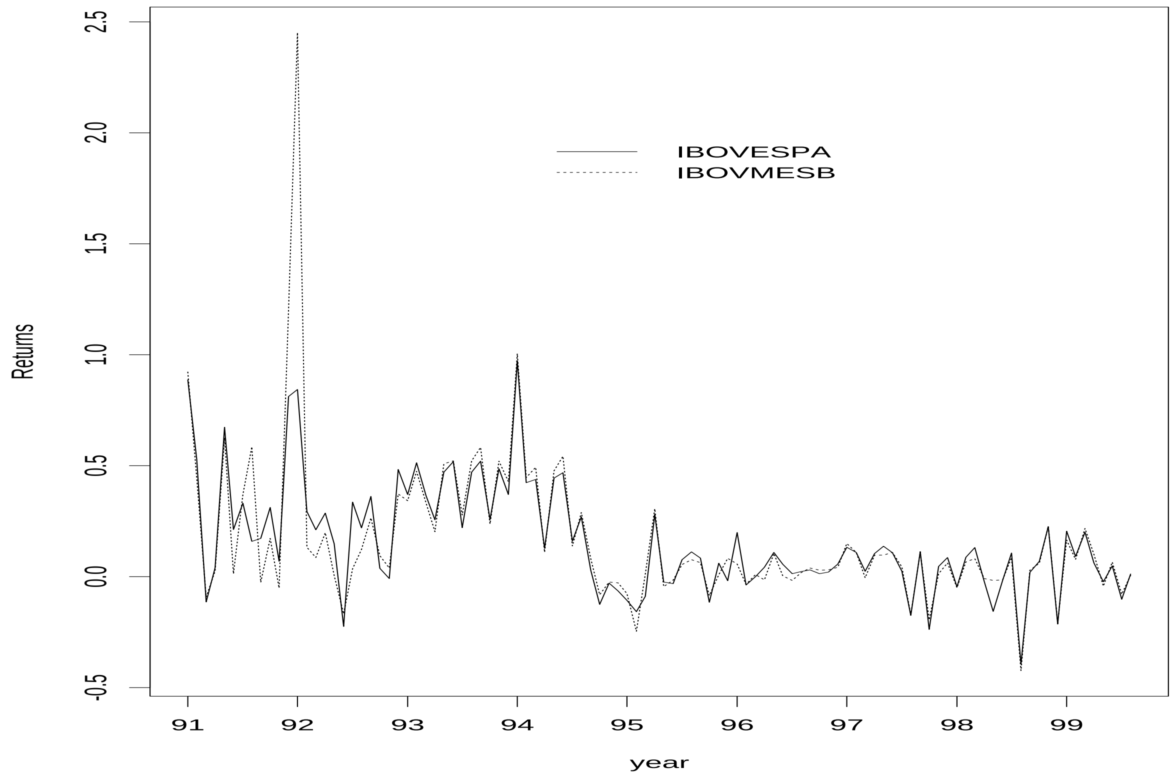
Figure 2:

algorithm
  read $X_1, \ldots, X_n$
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $f_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \int_{\Theta_{[ij]}} f_{[ij]}(\mathbf{X}_{[ij]}|\theta)\pi_{[ij]}(\theta)d\theta$
    $c^*_{[ij]} \leftarrow c_{[ij]} f_{[ij]}(\mathbf{X}_{[ij]})$
  end for
  for $k = 1$ **to** SAMPLES **do**
    generate U^k
  end for
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $r^*_{[ij]} \leftarrow$ proportion of samples such that
      $U^k_i = 0, U^k_{i+1} = \cdots = U^k_{j-1} = 1, U^k_j = 0$
  end for
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $\bar{X}_{[ij]} \leftarrow \frac{1}{j-i} \sum_{r=i+1}^{j} X_r$
    $m^*_{[ij]} \leftarrow \frac{(j-i)v_{[ij]}\bar{X}_{[ij]}}{(j-i)v_{[ij]}+1} + \frac{m_{[ij]}}{(j-i)v_{[ij]}+1}$
    $v^*_{[ij]} \leftarrow \frac{v_{[ij]}}{(j-i)v_{[ij]}+1}$
    $d^*_{[ij]} \leftarrow d_{[ij]} + j - i$
    $q_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \sum_{r=i+1}^{j} (X_r - \bar{X}_{[ij]})^2 + \frac{(j-i)(\bar{X}_{[ij]} - m_{[ij]})^2}{(j-i)v_{[ij]}+1}$
    $a^*_{[ij]} \leftarrow a_{[ij]} + q_{[ij]}(\mathbf{X}_{[ij]})$
  end for
  for $k = 1$ **to** n **do**
    $E(\mu_k|X_1, \ldots, X_n) \leftarrow \sum_0^{k-1} \sum_{j=k}^{n} r^*_{[ij]} m^*_{[ij]}$
    $E(\sigma^2_k|X_1, \ldots, X_n) \leftarrow \sum_{i=0}^{k-1} \sum_{j=k}^{n} r^*_{[ij]} \frac{a^*_{[ij]}}{d^*_{[ij]} - 2}$
  end for
  write $E(\mu_1), E(\sigma^2_1), \ldots, E(\mu_n), E(\sigma^2_n)$
end algorithm

Figure 3:

algorithm
  read $X_1, \ldots, X_n$
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $f_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \int_{\Theta_{[ij]}} f_{[ij]}(\mathbf{X}_{[ij]}|\theta)\pi_{[ij]}(\theta)d\theta$
    $c^*_{[ij]} \leftarrow c_{[ij]} f_{[ij]}(\mathbf{X}_{[ij]})$
  end for
  for $k = 1$ **to** SAMPLES **do**
    generate U^k
  end for
  for all $i, j \in \{0, \ldots, n\}$ **such that** $i < j$ **do**
    $\bar{X}_{[ij]} \leftarrow \frac{1}{j-i} \sum_{r=i+1}^{j} X_r$
    $m^*_{[ij]} \leftarrow \frac{(j-i)v_{[ij]}\bar{X}_{[ij]}}{(j-i)v_{[ij]}+1} + \frac{m_{[ij]}}{(j-i)v_{[ij]}+1}$
    $v^*_{[ij]} \leftarrow \frac{v_{[ij]}}{(j-i)v_{[ij]}+1}$
    $d^*_{[ij]} \leftarrow d_{[ij]} + j - i$
    $q_{[ij]}(\mathbf{X}_{[ij]}) \leftarrow \sum_{r=i+1}^{j}(X_r - \bar{X}_{[ij]})^2 + \frac{(j-i)(\bar{X}_{[ij]} - m_{[ij]})^2}{(j-i)v_{[ij]}+1}$
    $a^*_{[ij]} \leftarrow a_{[ij]} + q_{[ij]}(\mathbf{X}_{[ij]})$
  end for
  for $k = 1$ **to** n **do**
    $\mu\text{aux} \leftarrow 0$
    $\sigma^2_{\text{aux}} \leftarrow 0$
    for all i, j, l **such that**
      $U^l_i = 0, U^l_{i+1} = \cdots = U^l_k = \cdots = U^l_j = 1, U^k_{j+1} = 0$
        $\mu\text{aux} \leftarrow \mu\text{aux} + m^*_{[ij]}$
        $\sigma^2_{\text{aux}} \leftarrow \sigma^2_{\text{aux}} + \frac{a^*_{[ij]}}{(d^*_{[ij]}-2)}$
    end for
    $E(\mu_k|X_1, \ldots, X_n) \leftarrow \mu\text{aux}/\text{SAMPLES}$
    $E(\sigma^2_k|X_1, \ldots, X_n) \leftarrow \sigma^2_{\text{aux}}/\text{SAMPLES}$
  end for
  write $E(\mu_1), E(\sigma^2_1), \ldots, E(\mu_n), E(\sigma^2_n)$
end algorithm

Figure 4:



IBOVESPA
IBOVMESB
Returns
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
91
92
93
94
95
96
97
98
99
year

Figure 5:

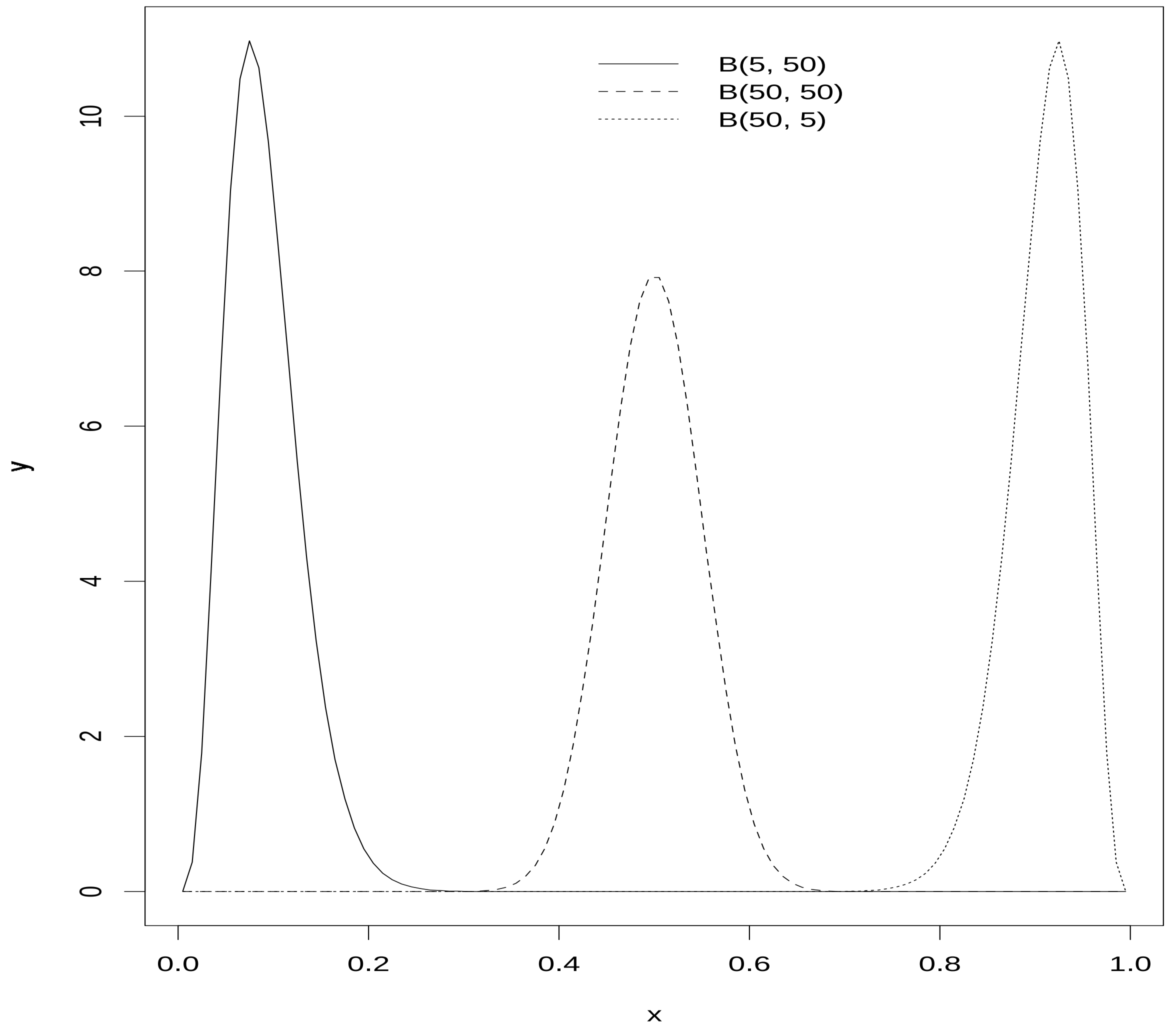


Figure 6:

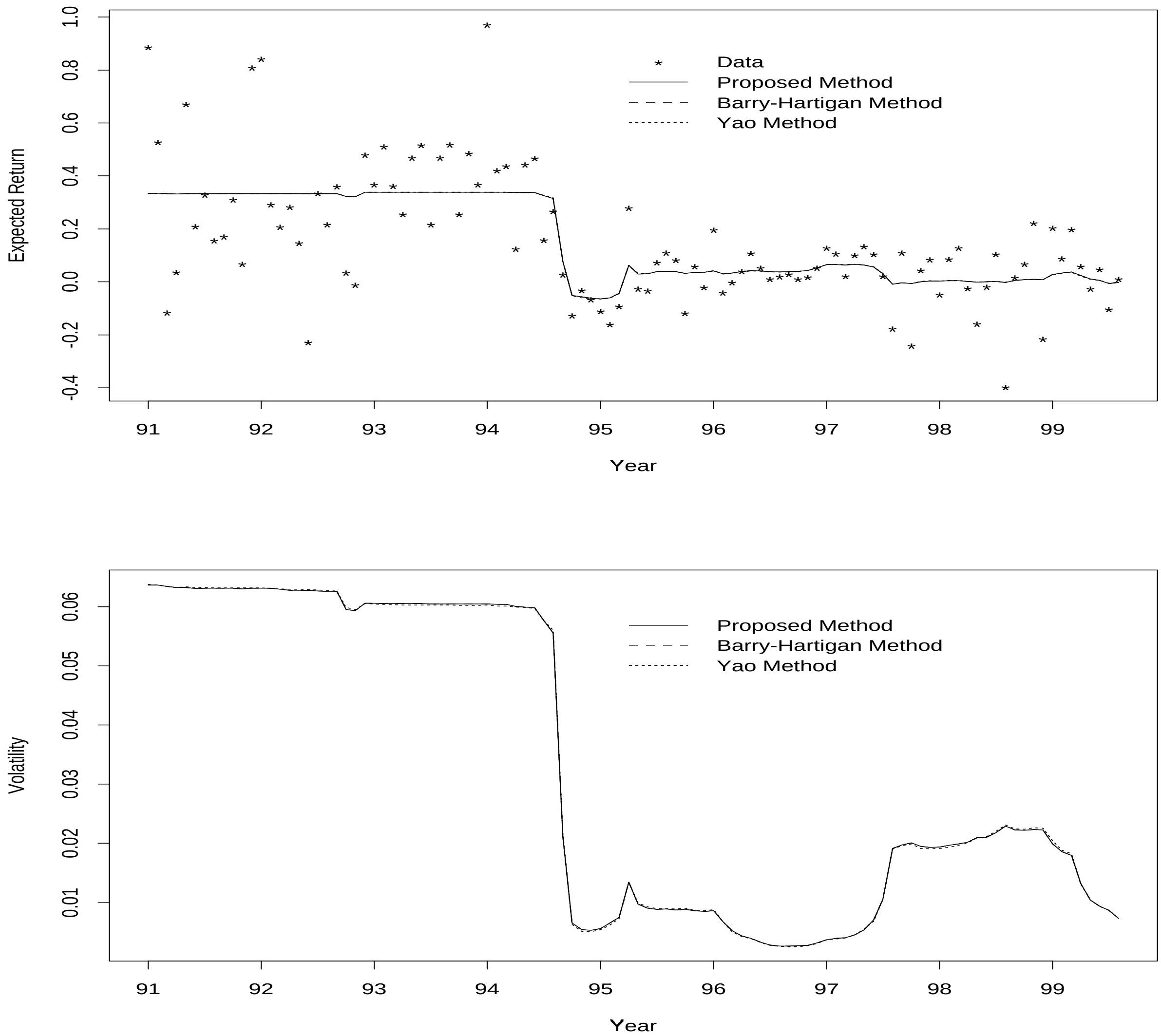
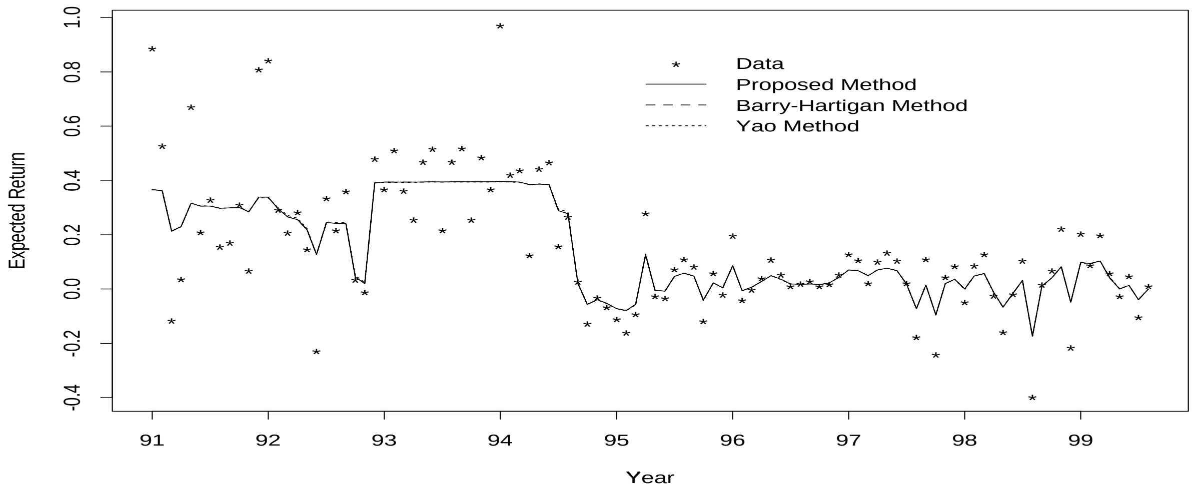
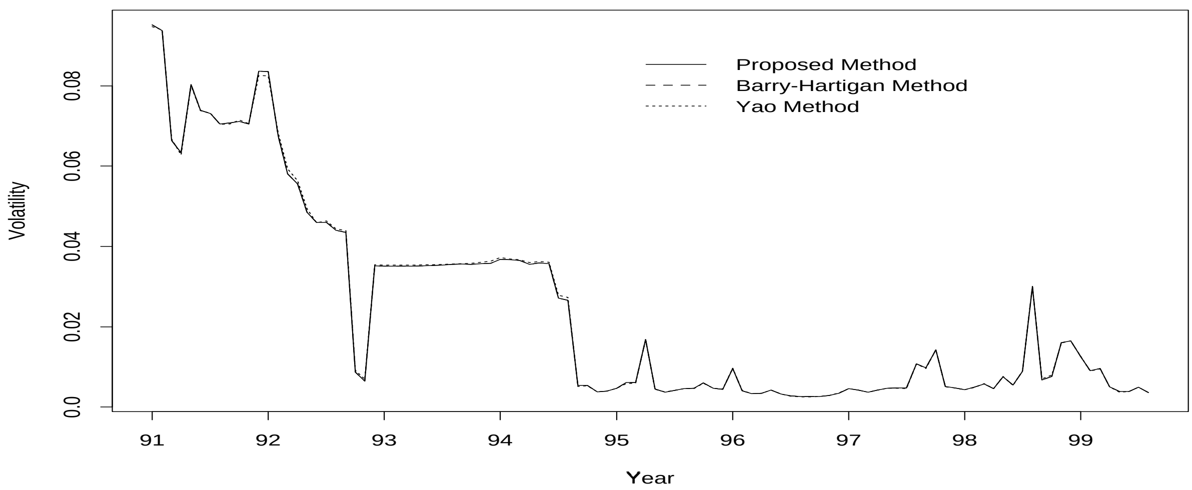

Expected Return
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Data
Proposed Method
Barry-Hartigan Method
Yao Method
91 92 93 94 95 96 97 98 99
Year
Volatility
0.01
0.02
0.03
0.04
0.05
0.06
Proposed Method
Barry-Hartigan Method
Yao Method
91 92 93 94 95 96 97 98 99
Year

Figure 7:

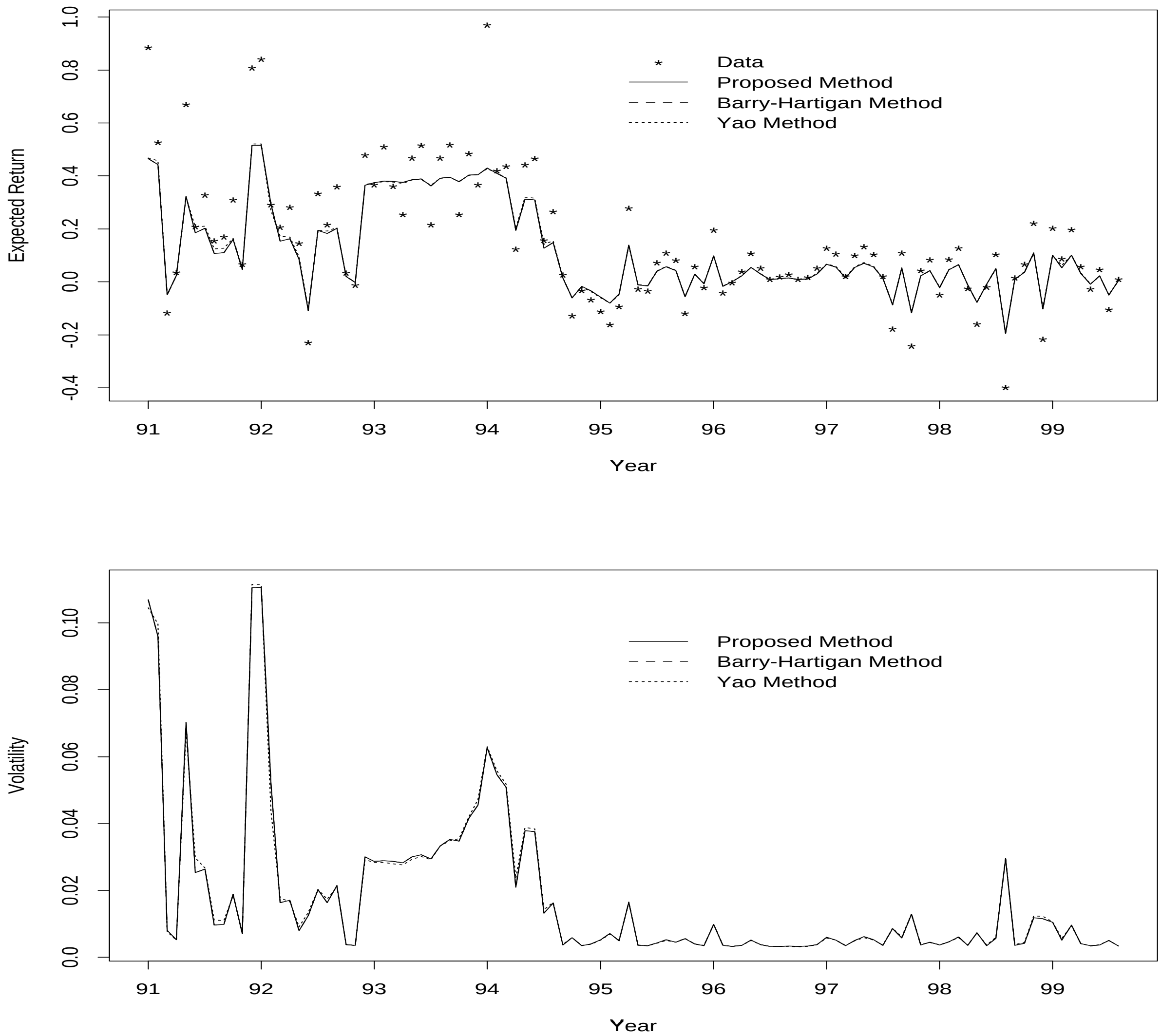


Expected Return
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
Data
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year



Volatility
0.08
0.06
0.04
0.02
0.0
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year

Figure 8:



Expected Return
Year
Data
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility

Figure 9:

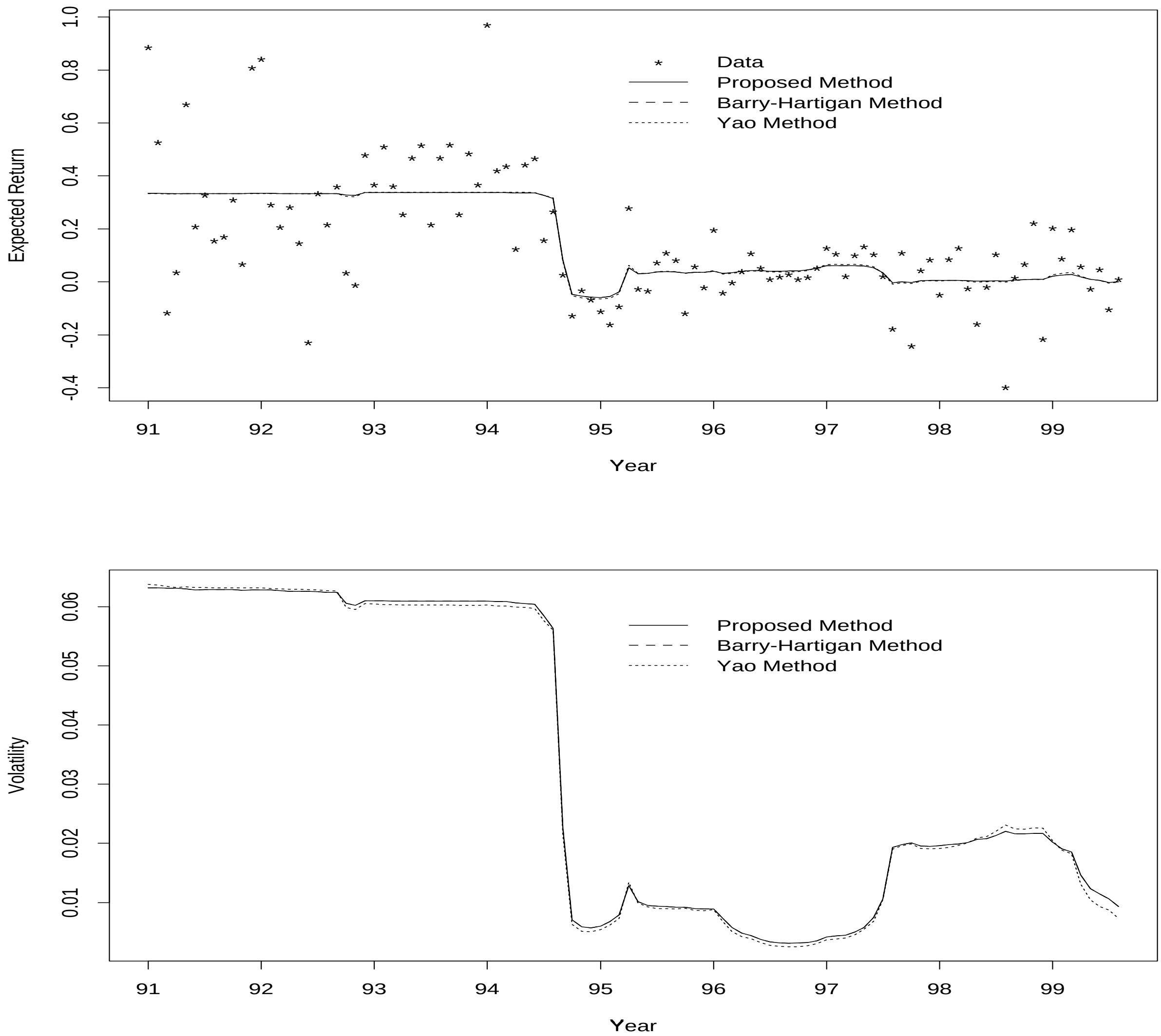

Data
Proposed Method
Barry-Hartigan Method
Yao Method
Expected Return
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
91
92
93
94
95
96
97
98
99
Year
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility
0.06
0.05
0.04
0.03
0.02
0.01
91
92
93
94
95
96
97
98
99
Year

Figure 10:

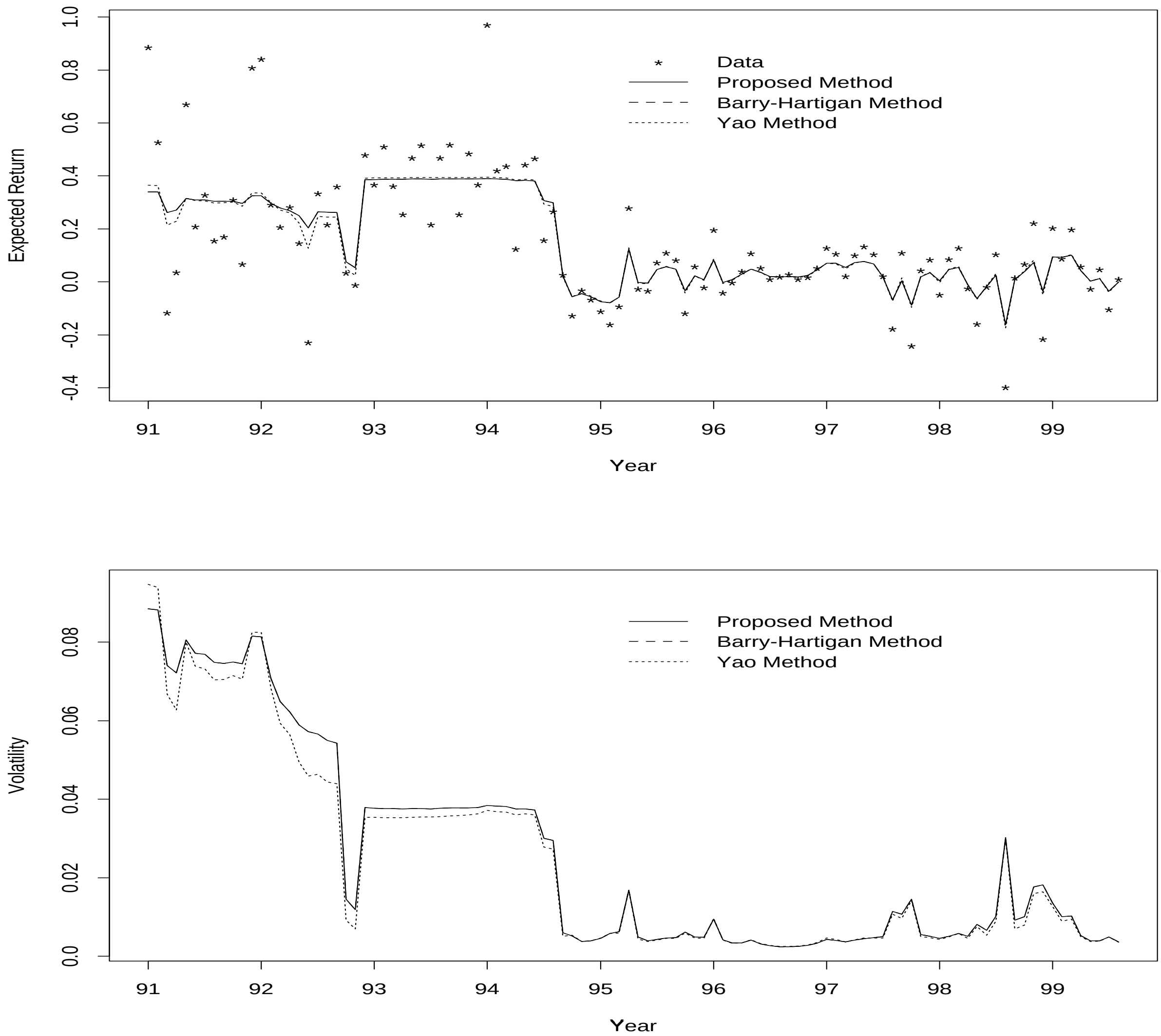

Data
Proposed Method
Barry-Hartigan Method
Yao Method
Expected Return
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
91
92
93
94
95
96
97
98
99
Year
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility
0.08
0.06
0.04
0.02
0.0
91
92
93
94
95
96
97
98
99
Year

Figure 11:

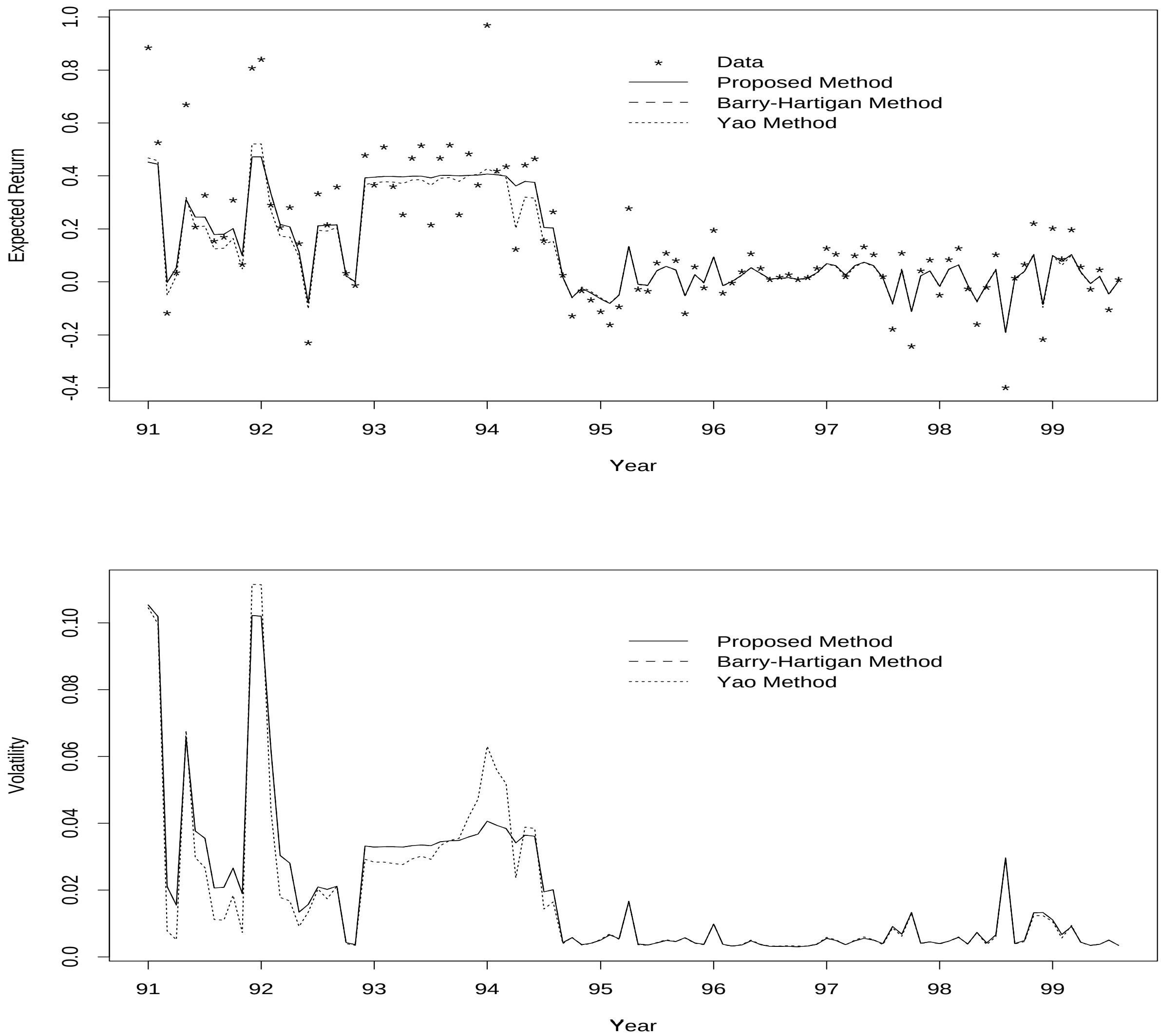

Expected Return
Year
Data
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility
Year
Proposed Method
Barry-Hartigan Method
Yao Method

Figure 12:

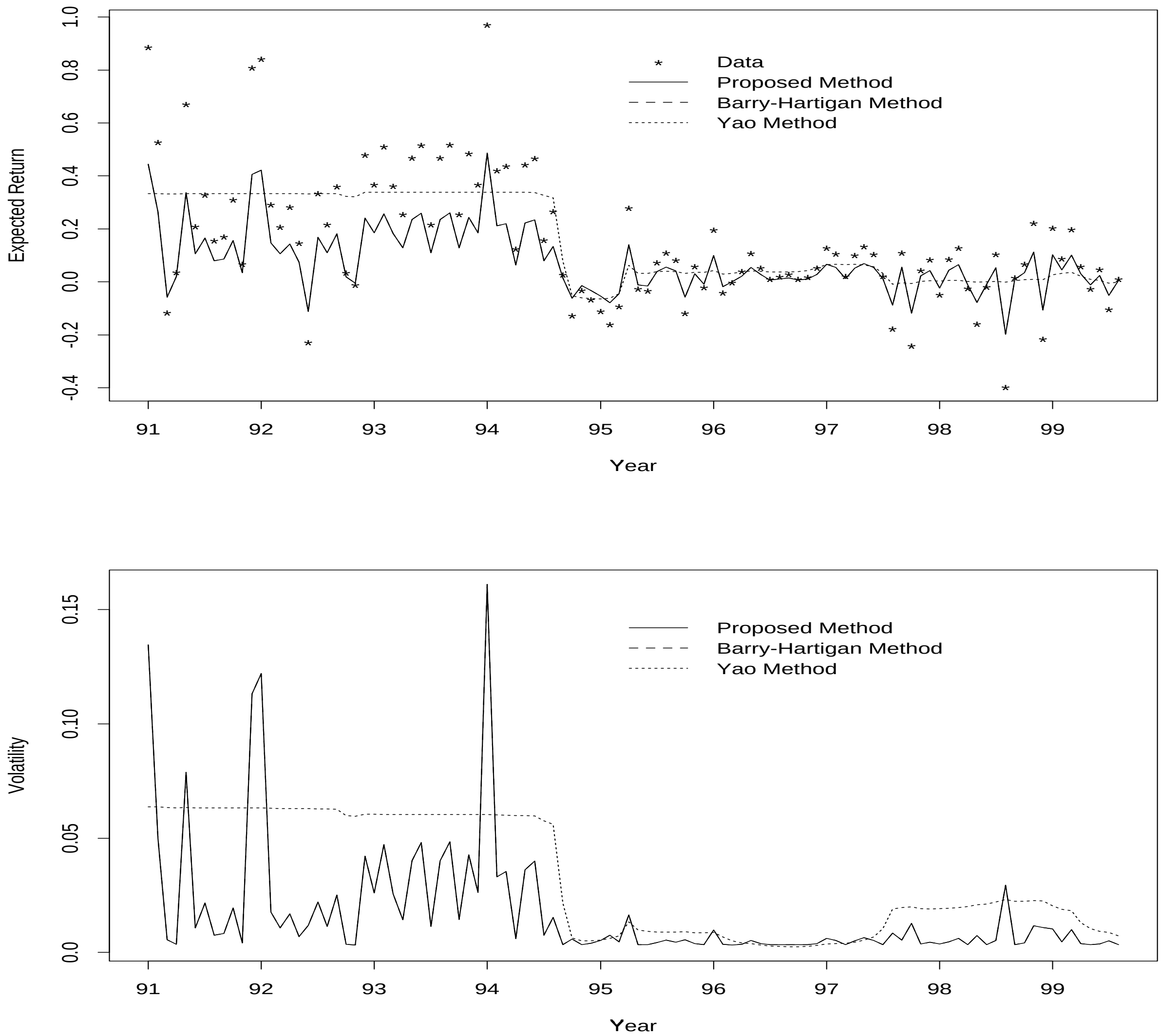

Expected Return
Year
Data
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility
Year
Proposed Method
Barry-Hartigan Method
Yao Method

Figure 13:

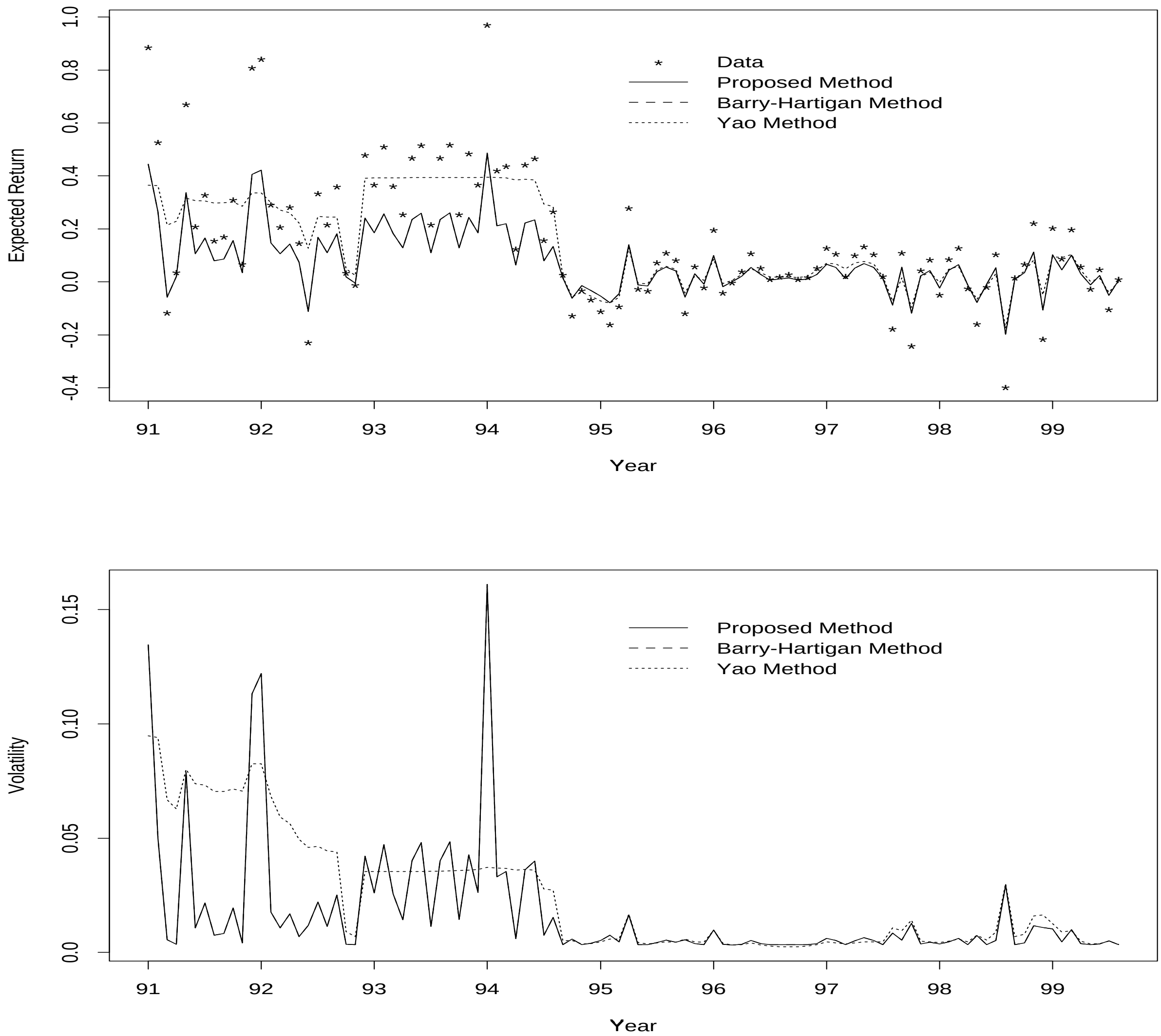

Expected Return
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
Data
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year
Volatility
0.15
0.10
0.05
0.0
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year

Figure 14:

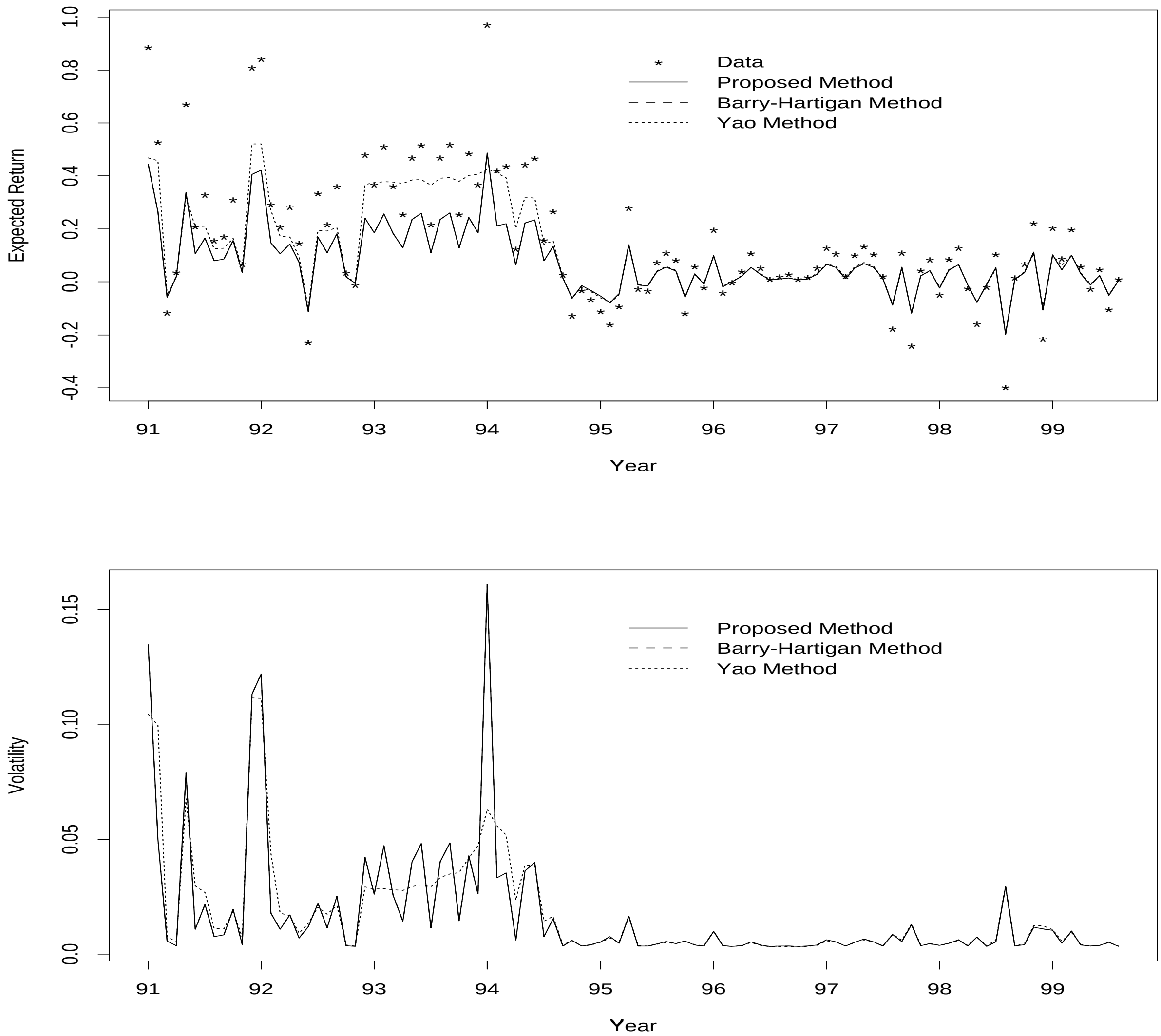
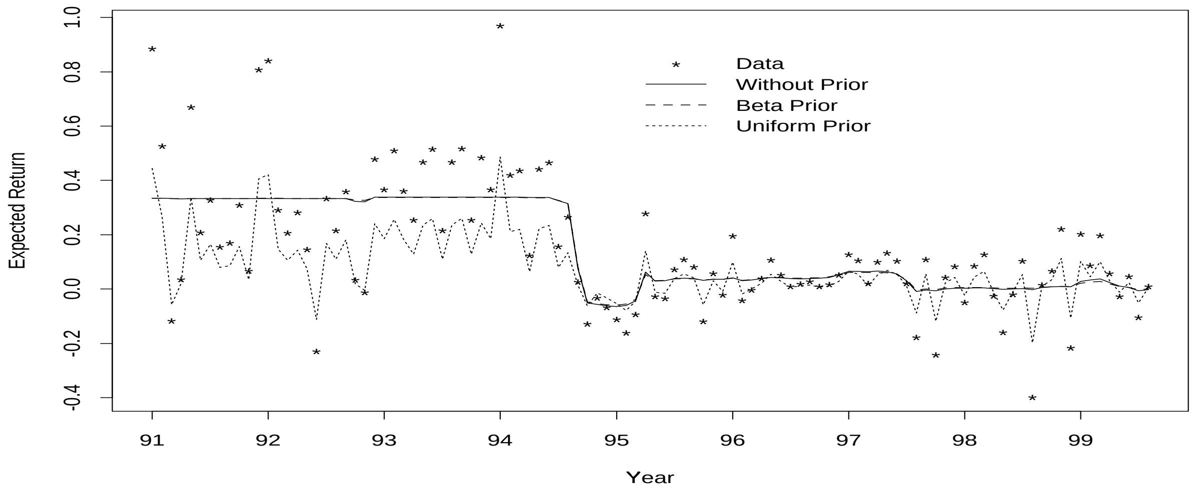

Expected Return
Year
Data
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility
Year
Proposed Method
Barry-Hartigan Method
Yao Method

Figure 15:


Expected Return
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Data
Without Prior
Beta Prior
Uniform Prior
91
92
93
94
95
96
97
98
99
Year

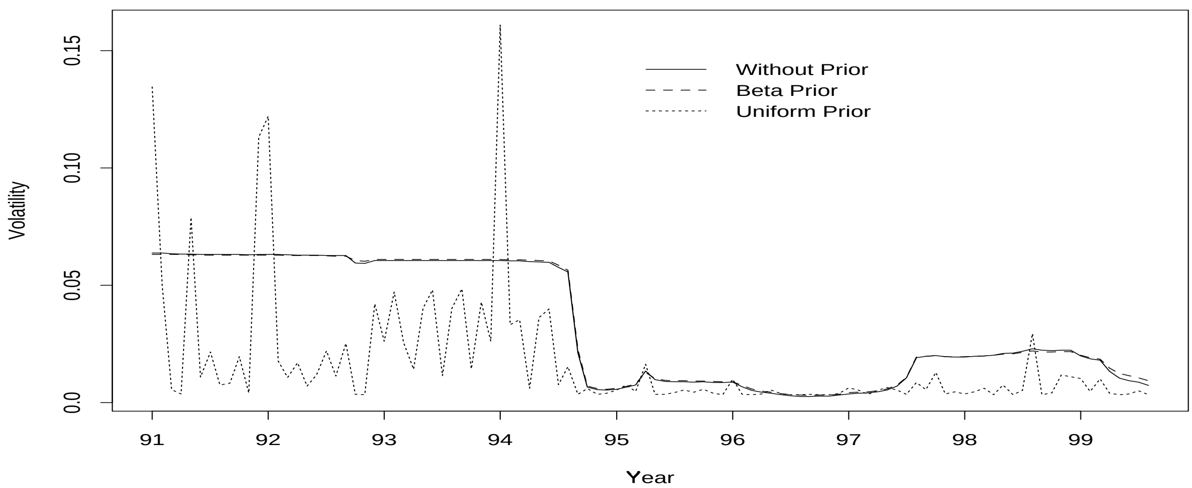

Volatility
0.0
0.05
0.10
0.15
Without Prior
Beta Prior
Uniform Prior
91
92
93
94
95
96
97
98
99
Year

Figure 16:

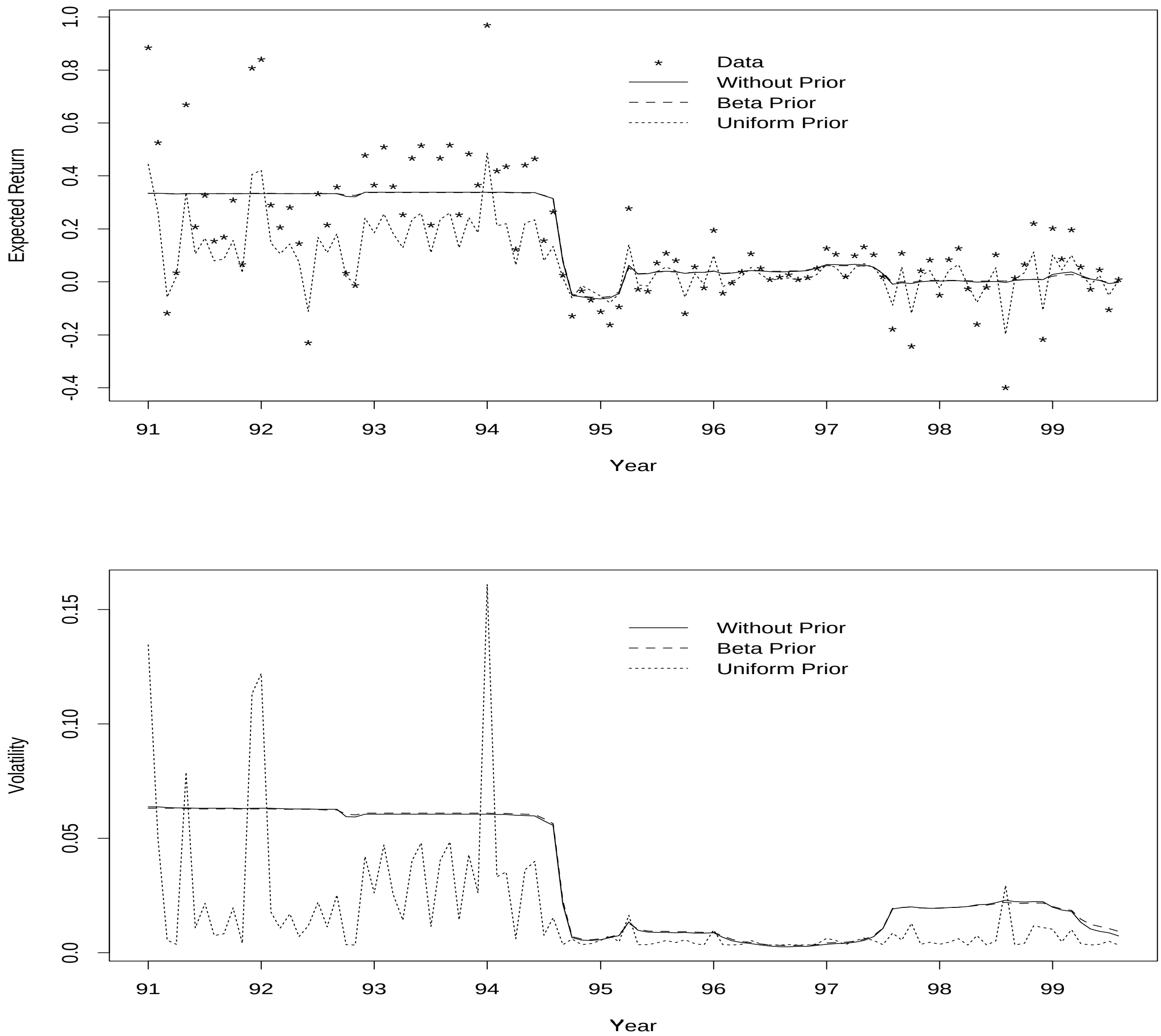
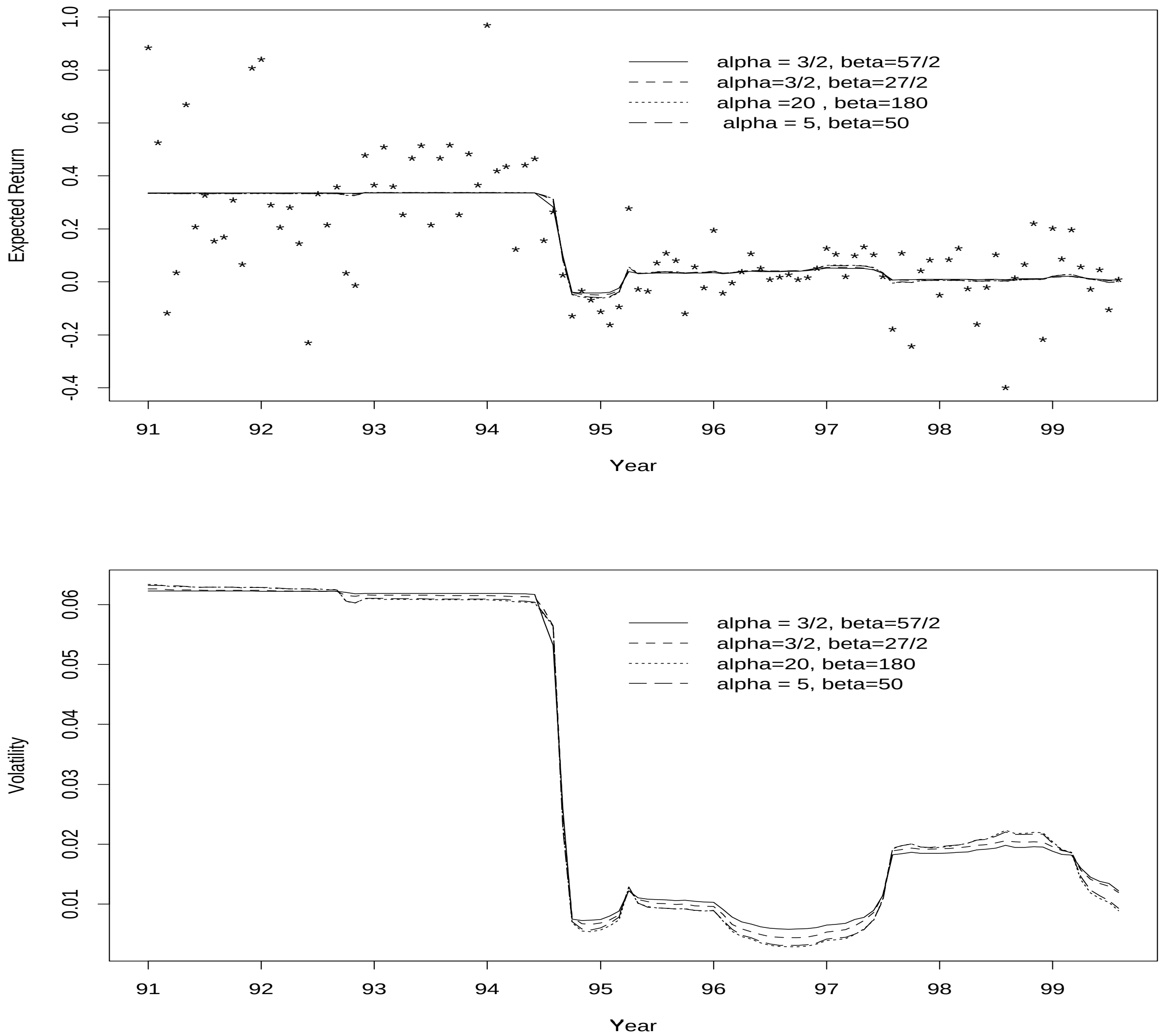

Data
Without Prior
Beta Prior
Uniform Prior
Expected Return
Year
Without Prior
Beta Prior
Uniform Prior
Volatility
Year

Figure 17:



Expected Return
alpha = 3/2, beta=57/2
alpha=3/2, beta=27/2
alpha =20 , beta=180
alpha = 5, beta=50
Year
Volatility
alpha = 3/2, beta=57/2
alpha=3/2, beta=27/2
alpha=20, beta=180
alpha = 5, beta=50
Year

Figure 18:

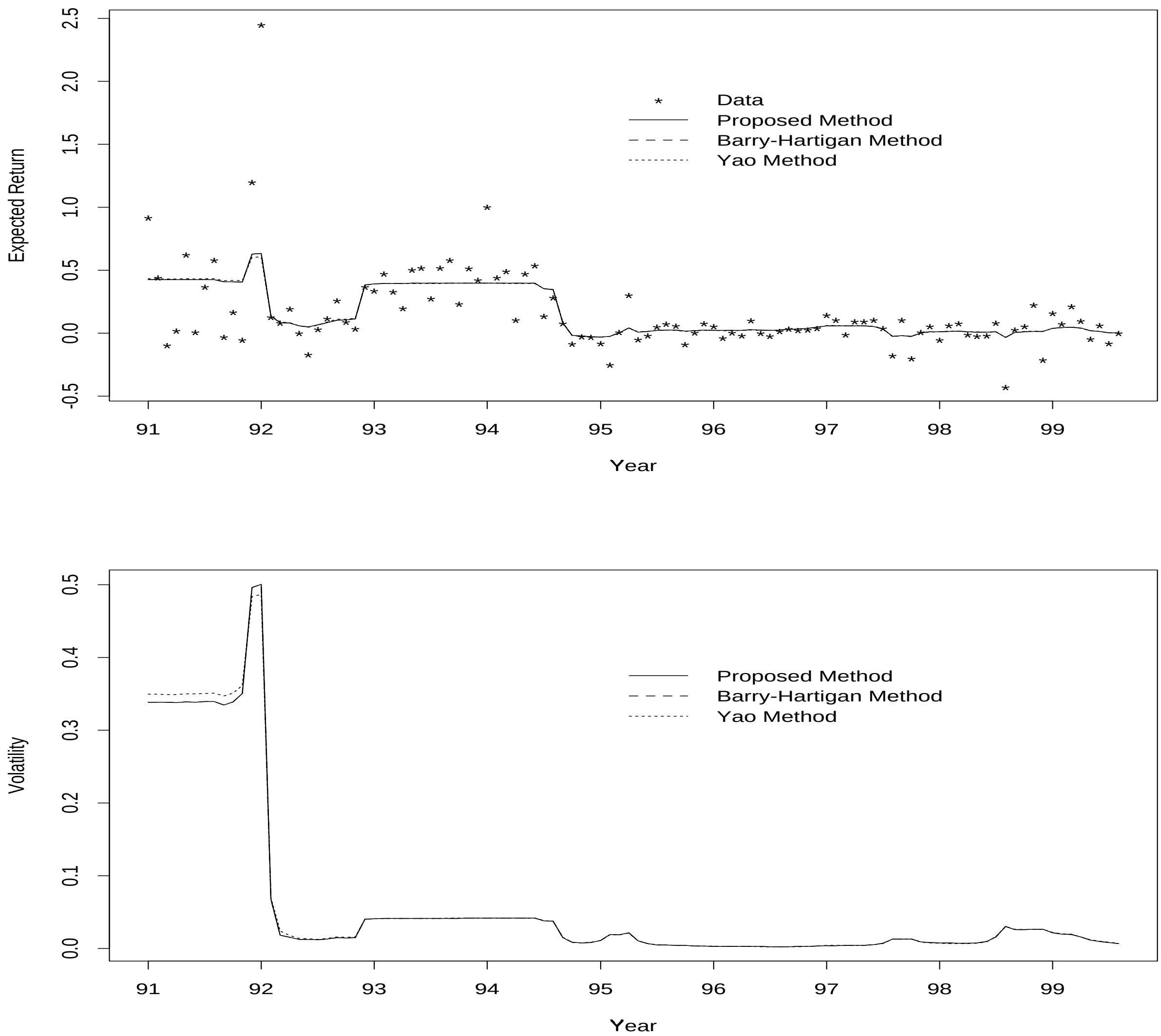

Expected Return
Year
Data
Proposed Method
Barry-Hartigan Method
Yao Method
Volatility
Year
Proposed Method
Barry-Hartigan Method
Yao Method

Figure 19:

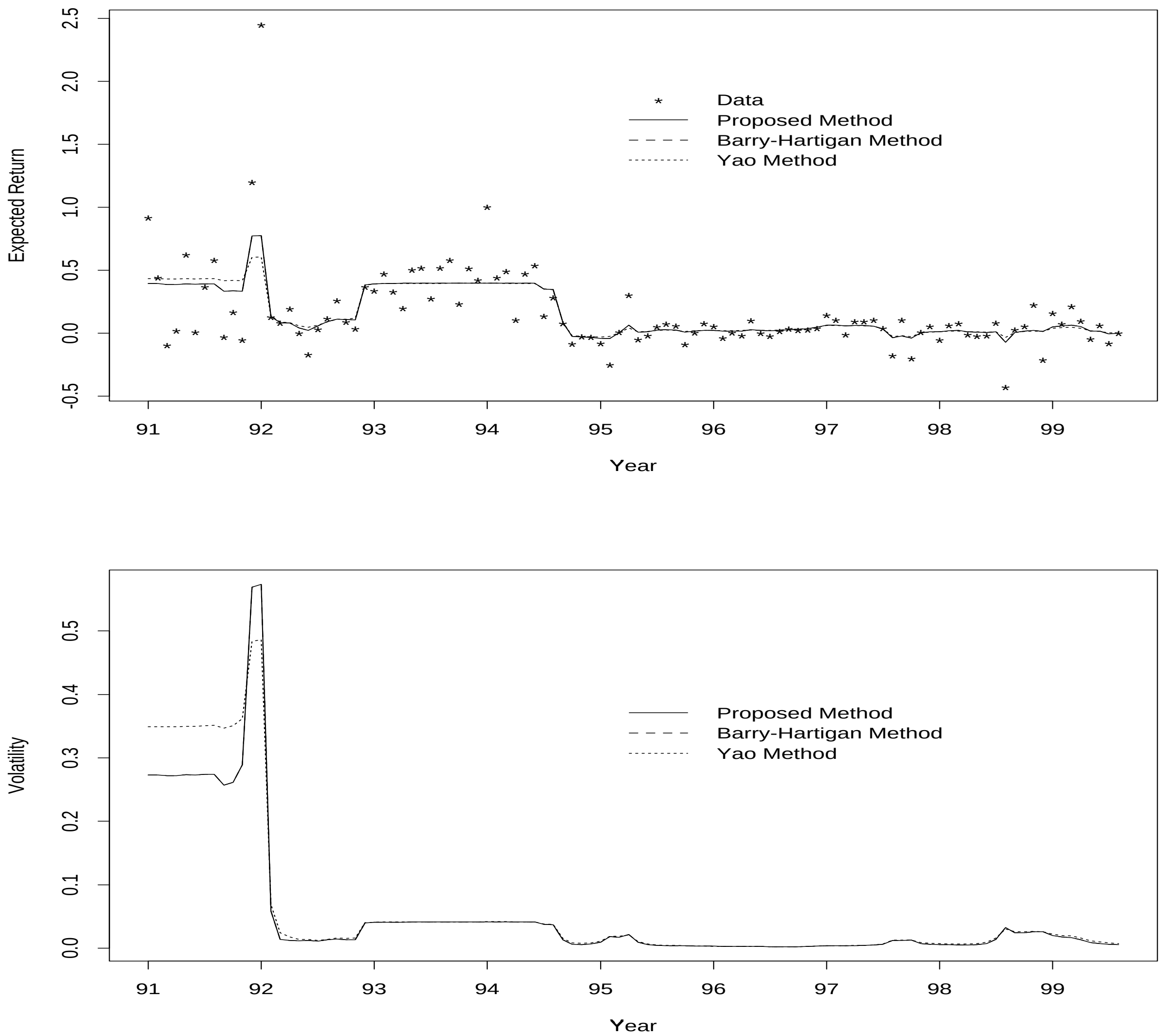
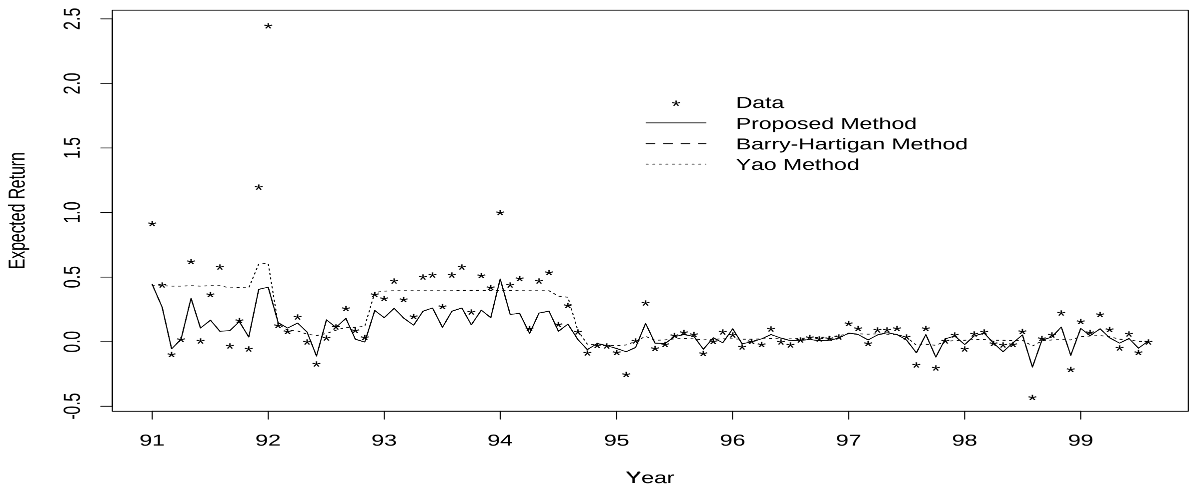

Expected Return
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
Data
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year
Volatility
0.0
0.1
0.2
0.3
0.4
0.5
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year

Figure 20:


Expected Return
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
Data
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year

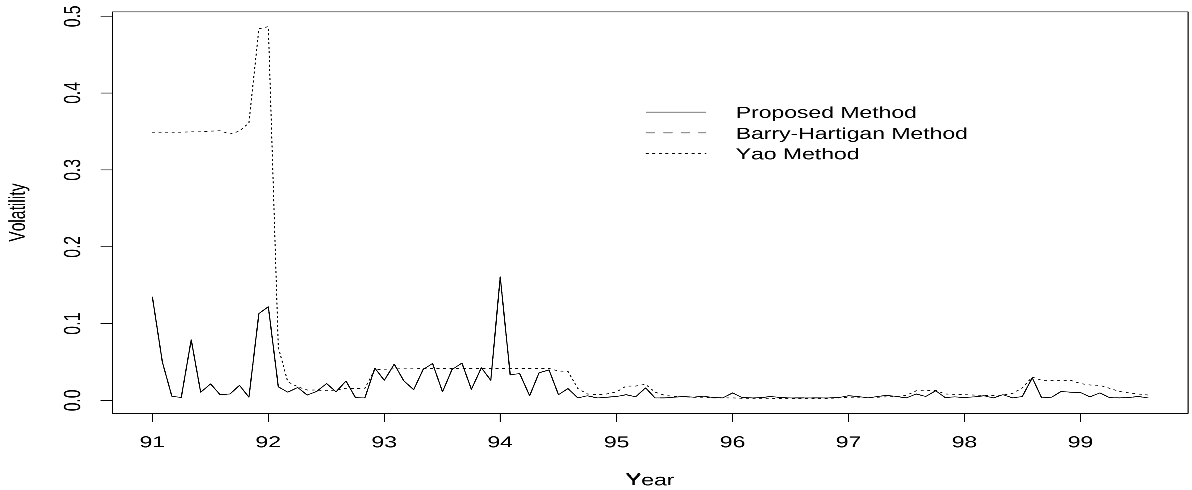

Volatility
0.0
0.1
0.2
0.3
0.4
0.5
Proposed Method
Barry-Hartigan Method
Yao Method
91
92
93
94
95
96
97
98
99
Year

Figure 21:

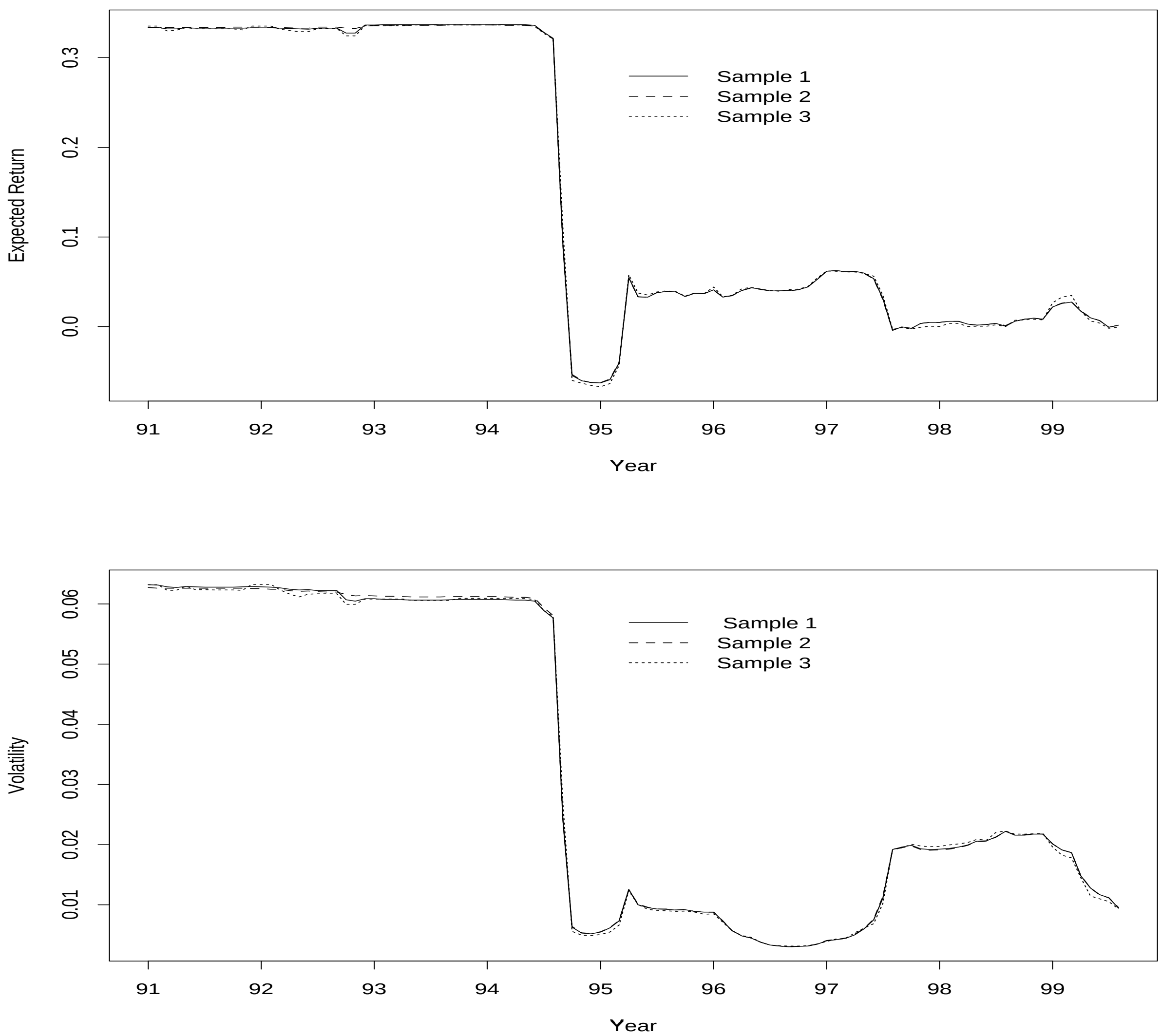

Sample 1
Sample 2
Sample 3
Expected Return
0.0
0.1
0.2
0.3
91
92
93
94
95
96
97
98
99
Year
Sample 1
Sample 2
Sample 3
Volatility
0.01
0.02
0.03
0.04
0.05
0.06
91
92
93
94
95
96
97
98
99
Year

Figure 22:

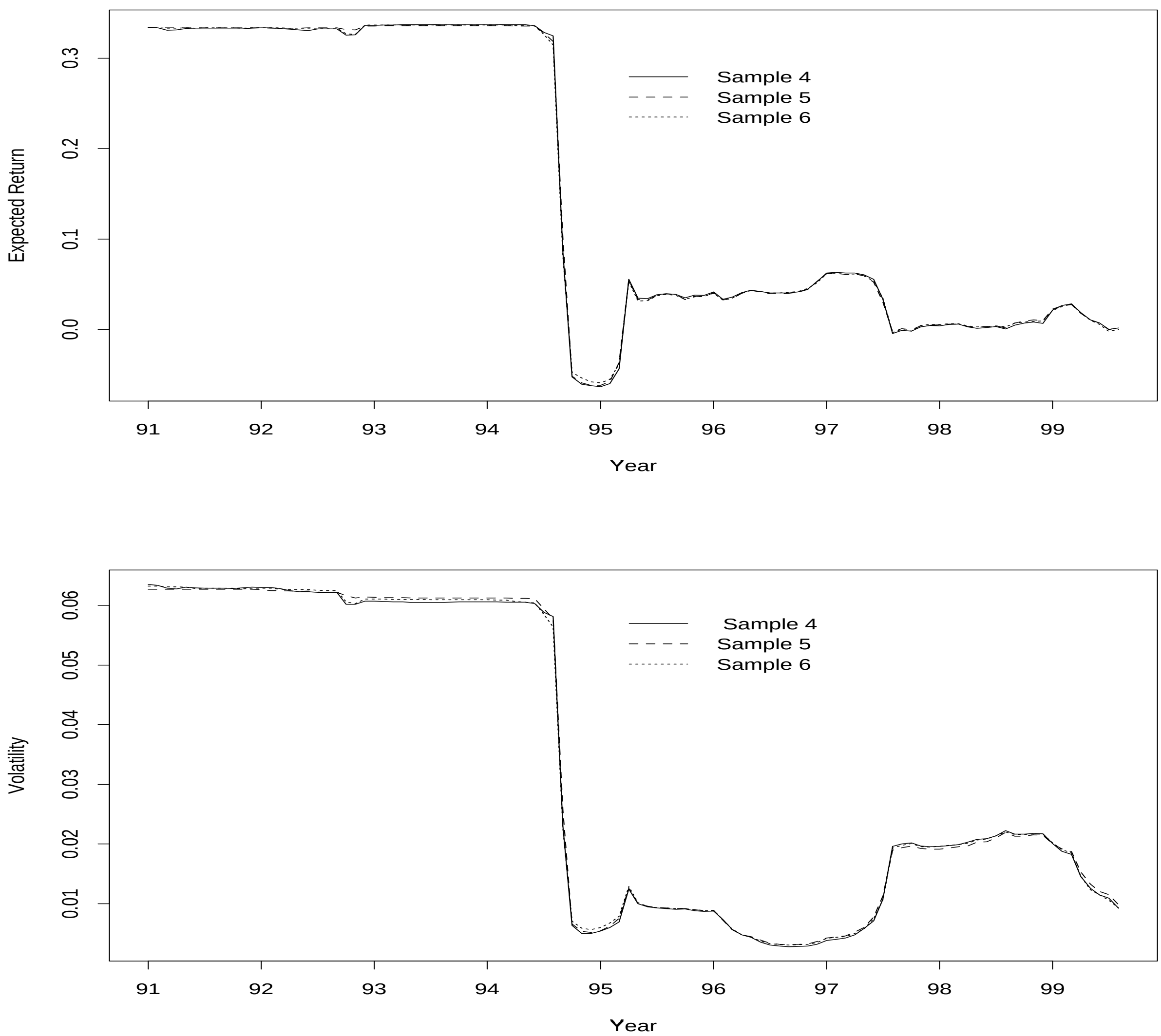

Expected Return
0.0
0.1
0.2
0.3
Sample 4
Sample 5
Sample 6
91
92
93
94
95
96
97
98
99
Year
Volatility
0.01
0.02
0.03
0.04
0.05
0.06
Sample 4
Sample 5
Sample 6
91
92
93
94
95
96
97
98
99
Year

Figure 23: