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# Asymptotic behavior of the ergodic measure of a Glauber+Kawasaky model*

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**Abstract**

We consider a interacting particle system, the Glauber +Kawasaki model. This model is the result of the combination of a fast stirring, the Kawasaki part, and a spin flip process, the Glauber part. This process has a Reaction-Diffusion equation as hydrodynamic limit, as is proven in [DFL]. The ergodicity of these dynamics in the presence of a metastable state (double well potential) was proven in [BPSV2]. In this article we prove that, in the limit, the expected value of each spin converges to the global minimizer of the potential. Also we prove decay of correlations of the ergodic measure.

## 1 Introduction

The main goal of this work is the discussion of a problem involving the long time behaviour for a class of interacting particle systems related to reaction diffusion equations.

The class of interacting particle systems which we study here has been proposed by De Masi, Ferrari and Lebowitz in 1986, as alternative models for reaction-diffusion systems. They are obtained as interacting particle systems on $\{-1,+1\}^{\mathbb{Z}^d}$, from the superposition of a Glauber type dynamics (spin flip, corresponding to reactive part) and a stirring one (also called Kawasaki dynamics at infinite temperature), corresponding to the diffusive part, and which is speeded by a factor, say $\epsilon^{-2}$. The kinetic limit which here corresponds to the hydrodynamical one involves the simultaneous change of space scaling in the diffusive limit for the stirring, to provide the macro scale (macro= $\epsilon$ micro). Under such limit, the macroscopic description is verified, being given by a reaction diffusion equation of the form

$$\partial_t m = \Delta m + F(m) \tag{1}$$

where $m(r,t) \in \mathbb{R}$ represents magnetization or density, the force term $F(\cdot)$ being determined by the Glauber rates. For example, given any polynomial $F(\cdot)$ we may choose

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finite range spin flip rates which lead to the above equation (of course there are inumerous choices, though the behaviour should be similar). That is, the empirical magnetization, or the density of particles in small boxes, converges in probability to the solution of the reaction diffusion equation. Indeed one knows more: strong forms of propagation of chaos (i.e. asymptotic independence of different spins) have been proven, initially in De Masi et al.(86), [DFL], and later in sharper forms by several authors, see A.De Masi, E.Presutti (91) [DP]. At the level of propagation of chaos, more general systems, with $m(r,t)$ taking values in $\mathbb{R}^k$ are treated just in the same way, though we restrict to the case $k = 1$ and so there is no loss in assuming $F(m) = -V'(m)$ in the above equation.

In spite of the validity of propagation of chaos, which corresponds to the infinite temperature situation, for any fixed value of the scale parameter $\epsilon$ the product measures are not (except for trivial cases) invariant for the system. This is clear due to the presence of a spin flip dynamics which corresponds to a non-product Gibbs measure, at finite temperature. Thus, the problem of looking at what happens at times which tend to infinity as $\epsilon \to 0$, or even for fixed, but arbitrarily small positive values of $\epsilon$ is an important one. These questions could be roughly stated as: what happens beyond the "hydrodynamical scaling"? To these set belong problems of clear importance in the description of non-equilibrium behaviour such as phase separations. The simpler way is to consider $V(\cdot)$ as a double well and to ask about the escape from the unstable homogeneous fixed point of the reaction-diffusion equation, i.e., the local maximun in $V(\cdot)$. Physically it corresponds to the situation of a gas which from a high temperature is suddenly cooled below the critical one, so that it becomes unstable, with the formation of liquid droplets having the coexistence of regions of vapor and others of liquid, a state which does not correspond to a pure phase, but is a mixture of such. There are several works in this direction, describing to a certain extend the onset of phase separation in this class of models, i.e. when does it escape from the homogeneous unstable point, how are the incipient clusters corresponding to $m_+$ and $m_-$ after the escape, and how do they evolve. (See e.g. Calderoni et al. (89) [C], Schonmann (98) [Sch])

Going further, one is naturally lead to a full multi-scale behaviour, and the complete description gets quite hard. But, going to the other extremal situation, when $\epsilon > 0$ is fixed and time tends to infinity, one is lead to investigate the ergodic behaviour of such systems. We refer to Liggett,(2000), [L2] for a discution of this. For a class of rates leading to a single well potential, the ergodicity for any $d \geq 1$, has been proven by Brassesco et. al. (2000) [BPSV1]. The same authors also proved ergodicity in the case of a double well potential provided $V(\cdot)$ which has a unique minimizer, cf. Brassesco et. al. (2000) [BPSV2]. We also refer to the article of Durrett and Newhauser (94) [DN], where a large classes of spin flip rates were considered, with the characteristic of having a trapping configuration (e.g. the contact process). In such case phase transition might occur, for the same reason which leads to the existence of an invariant measure with magnetization different from the trapping configuration, i.e. is related to the minimizer of the potential. Also for a finite configurations space the system has a metastable behavior and the asymptotic of the exit time from a neighborhood of the metastable state is analyzed in [H].

In this work, we complete the features related to the unique invariant measure of the process, in the double well potential case. First, we prove that in the limit as $\epsilon$ goes to zero the espected value of a coordinate is $m_+$, the global minima of the potential. For this we complete the percolation argument in the proof of ergodicity in Brassesco et. al. [BPSV2], from which follows easely the result. As a consequence of the exponential convergence to the invariant measure as proven in Brassesco et. al. [BPSV2], and a clasical result of Liggett, we get the exponential decay of correlations of two spins. The obtained lower bound for the decay depends only on the distance between the coordinates times a factor $\epsilon^2$.

## 2 Definitions and results

We consider, for each $\epsilon > 0$, a Markov process $\{\sigma_t^{(\epsilon)}\}_{t\geq 0}$, $\sigma_t^{(\epsilon)} \in \Omega = \{-1, 1\}^{\mathbb{Z}}$, where we let $\mathbb{Z}$ be the set of integers. The generator of the process (Glauber+Kawasaki process) is:

$$L_\mu^{(\epsilon)} = \epsilon^{-2} L_0 + L_\mu, \tag{2}$$

where $\epsilon^{-2} L_0$ is the accelerated stirring process defined by:

$$L_0 f(\sigma) = \sum_{\substack{|x-y|=1,\\ x,y \in \mathbb{Z}^d}} [f(\sigma^{x,y}) - f(\sigma)], \tag{3}$$

where $f$ is a cilindric function and $\sigma \in \Omega$. Now, we take $\mu > 0$, and $\gamma \in (\frac{1}{}$

$$\frac{}{2}\sigma(x), \tag{5}$$

and $c_0(x, \sigma)$, the intensity when $\mu = 0$ is:

$$c_0(x,\sigma) = \frac{1}{d}\sum_{i=1}^{d} \{1 - \gamma\sigma(x)[\sigma(x-e_i) + \sigma(x+e_i)] + \gamma^2\sigma(x-e_i)\sigma(x+e_i)\}, \tag{6}$$

where $e_i$ is the i-th coordinate unit vector. We restrict $\mu$ to $0 < \mu < 2(1-\gamma)^2$ for $c_\mu(x,\sigma)$ be positive.

For each $\epsilon > 0$, $\{\sigma_t^{(\epsilon)}\}_{t\geq 0}$ is a Markov process. We abreviate $\sigma_t = \sigma_t^{(\epsilon)} \in \Omega$. If $\nu$ is a probability on $\Omega$ (resp. a single configuration $\sigma$), we will denote by $E_\nu^{(\epsilon)}$, (resp. by $E_\sigma^{(\epsilon)}$) the expectation of the above process starting with law $\nu$ (resp. from the configuration $\sigma$). Also from [DFL] we have that, as $\epsilon \to 0$ the process converges to the solution of the reaction-diffusion equation:

$$\frac{\partial m}{\partial t} = \Delta m - V_\mu'(m)\,. \tag{7}$$

where :



1) For the

$$-V_\mu' = \mathbb{E}_{\nu_m}\left(-2\sigma\left(0\right)c_\mu\left(0,\sigma\right)\right) = -\alpha m - \beta m^3 + \mu, \tag{8}$$

$\nu_m$ denoting the Bernoulli measure on $\Omega$, with average $m$ and

$$\alpha = 2(1-2\gamma), \beta = 2\gamma^2. \tag{9}$$

The polynomial $V_\mu'$ is the derivative of the double well potential $V_\mu$. In [BPSV2], it was proved that for $\epsilon$ small enough the process is ergodic:

**Theorem 2.1** *For any $d \geq 1$, $\gamma < 1$ and $\mu > 0$, there is a $\epsilon_{\gamma,\mu} > 0$ so that for any $\epsilon < \epsilon_{\gamma,\mu}$, the generator $L_\mu^{(\epsilon)}$ has a unique invariant measure $\mu_\epsilon$.*

Observe that (8) is not ergodic (in the sense that it has two invariant solutions), indeed, if we call $m_\pm$ the two points of local minimum of $V_\mu$, the states $m_\pm(r) = m_\pm$ are stationary solutions of (8), locally stable. Also we have $V_\mu(m_+) < V_\mu(m_-)$.

The main part in the proof of the last theorem is to study the discrepancies $\sigma_t^+ - \sigma_t^-$ between the configurations starting from $\mathbf{1}$ (ie. with all the spins equal to 1) and $-\mathbf{1}$ (resp. equal to -1). Using the graphical construction (see Appendix), we have that $\sigma_t^+ \geq \sigma_t^-$, since the process is attractive. The authors prove that for $\epsilon$ small enough there exists $c(\epsilon)$ and $\lambda > 0$ such that:

$$\mathbb{E}^{(\epsilon)}_{\mathbf{1},-\mathbf{1}}\left(\sigma^+(0,t) - \sigma^-(0,t)\right) \leq c(\epsilon)e^{-\lambda t}. \tag{10}$$

Recall that the process $\sigma_t^{(\epsilon)}$ converges, as $\epsilon$ goes to zero, to the reaction-diffusion equation (8), and that this equation has two invariant solutions,$m_-$, $m_+$. The last constant, $c(\epsilon)$, is not optimal, it depends on $\epsilon$, and is related to the tunneling event to pass from configurations with magnetization in the basin of $m_-$ to the configurations in the other one, $m_+$.

Regarding this, we can ask about the behavior of $\mu_\epsilon$ as $\epsilon$ goes to zero. We prove that:

**Theorem 2.2** *For $\mu_\epsilon$ the invariant measure for the process $\sigma_t^{(\epsilon)}$ we have that:*

$$\lim_{\epsilon\to 0}\mu_\epsilon(\sigma(0)) = m_+. \tag{11}$$

We are also interested in studying the decay of correlations of the invariant measure $\mu_\epsilon$. Concernig this we prove that:

**Theorem 2.3** *For $\epsilon$ small enough, and for all $x \in \mathbb{Z}^d$ it holds:*

$$|\mu_\epsilon(\sigma(0)\sigma(x)) - \mu_\epsilon(\sigma(0))\mu_\epsilon(\sigma(x))| \leq c(\epsilon)e^{-c\epsilon^2|x|}. \tag{12}$$

## 2.1 Proof of Theorem 2.2

We need to prove that:

**Lemma 2.1** $\limsup_{\epsilon\to 0}\mu_\epsilon(\sigma(0)) \leq m_+$*;*

and

**Lemma 2.2** $\liminf_{\epsilon\to 0}\mu_\epsilon(\sigma(0)) \geq m_+$.

The first Lemma is easy to prove. Fix $\epsilon > 0$ small enough. Since $\mu_\epsilon$ is ergodic: $\mathbb{E}^{(\epsilon)}_{\mathbf{1}}(\sigma_t(0)) \to \mu_\epsilon(\sigma(0))$ (resp. $\mathbb{E}^{(\epsilon)}_{-\mathbf{1}}(\sigma_t(0)) \to \mu_\epsilon(\sigma(0))$), as $t \to \infty$.

So that, using the atractiveness and the time invariance of $\mu_\epsilon$ we have that, for all $t > 0$:

$$\mathbb{E}^{(\epsilon)}_{-\mathbf{1}}(\sigma_t(0)) \leq \mu_\epsilon(\sigma(0)) \leq \mathbb{E}^{(\epsilon)}_{\mathbf{1}}(\sigma_t(0)).$$

Let $\delta > 0$, using the proximity of the process to the solution of the reaction -difussion equation, for each $t$; and also the proximity of the solution which starts on $\mathbf{1}$, to $m_+$ for $t$ large enough, then there exists $t_0(\epsilon,\delta)$, such that for all $t \geq t_0$:

$$|\mathbb{E}^{(\epsilon)}_{\mathbf{1}}(\sigma_t(0)) - m_+| \leq \delta,$$

then:

$$\begin{aligned}\mu_\epsilon(\sigma(0)) &\leq \mathbb{E}^{(\epsilon)}_{\mathbf{1}}(\sigma_t(0)) \\ &\leq \delta + m_+.\end{aligned}$$

Since $\delta$ is arbitrary the Lemma 2.1 is proven.

**Proof (Lemma 2.2)**

Let $\Delta(n)$ the cube centered at zero in $\mathbb{Z}^d$ with side $n$ , let $|\Delta(n)|$ be the number of sites $x \in \mathbb{Z}^d$ which are contained in $\Delta(n)$. Define the magnetization:

$$M_n(\sigma,t) := \frac{1}{|\Delta(n)|}\sum_{x\in\Delta(n)}\sigma_t(x).$$

If this average is over a cube centered at $x$, then we write $M^x_n(\sigma,t)$, (when is clear from the context we omit the superscript $x$).

As before, fix $\epsilon > 0$ small enough. Since the process is translation invariant and $\mathbb{E}^{(\epsilon)}_{-\mathbf{1}}(\sigma_t(0)) \leq \mu_\epsilon(\sigma(0))$ we have that:

$$\mathbb{E}^{(\epsilon)}_{-\mathbf{1}}(M_n(\sigma,t) - m_+) + m_+ \leq \mu_\epsilon(\sigma(0)). \tag{13}$$

For $\xi > 0$ observe that:

$$\begin{aligned}\mathbb{E}^{(\epsilon)}_{-\mathbf{1}}(M_n(\sigma,t) - m_+) &\geq \mathbb{E}^{(\epsilon)}_{-\mathbf{1}}[(M_n(\sigma,t) - m_+)\mathbf{1}_{\{(M_n(\sigma,t)-m_+)<-\xi\}}] \\ &- \xi P^{(\epsilon)}_{-\mathbf{1}}[(M_n(\sigma,t) - m_+) > -\xi] \\ &\geq (-1 - m_+)P^{(\epsilon)}_{-\mathbf{1}}[(M_n(\sigma,t) - m_+) < -\xi] - \xi \\ &\geq (-1 - m_+)P^{(\epsilon)}_{-\mathbf{1}}[|M_n(\sigma,t) - m_+| > \xi] - \xi. \end{aligned} \tag{14}$$

So it suffices to obtain an upper bound for the probability in the last inequality, which goes to zero as $\epsilon$ tends to zero. This is the content of the next proposition.

We modify the oriented percolation argument in the proof of Proposition 5.1 of [BPSV2], to get the upper bound that we mentioned before, and complete the proof of the former proposition . First we present some definitions.

Let $\mathcal{D}^{(\ell_i)}$, $i = 1, 2$ denote two partitions of $\mathbb{Z}^d$ into cubes $\Delta(\ell_i)$, where:

$$\ell_1 \approx \epsilon^{-1/10}, \quad \ell_2 \approx \epsilon^{-1}|\log \epsilon|. \tag{15}$$

We will suppose that each cube $\Delta(\ell_2)$ is the union of cubes $\Delta(\ell_1)$ so that $\mathcal{D}^{(\ell_1)}$ is finer than $\mathcal{D}^{(\ell_2)}$. For each $x \in \mathbb{Z}^d$ we define:

$$\eta_\xi(x,t) = \begin{cases} 1 & \text{if } |M^x_{(\ell_1)}(\sigma,t) - m_+| < \xi; \\ 0 & \text{otherwise.} \end{cases}$$

The pair $(i,k)$ of $\mathbb{Z}^d \times \mathbb{N}$ will represent the cube $\Delta_i(\ell_2)$, element of the partition $\mathcal{D}^{(\ell_2)}$, which contains $i\ell_2$ and the time $kT|\log \epsilon|$, where $T$ will be apropiately chosen. Finally define the variable $\chi(i,k)$ with values $G$ ($G$ for good) and $B$ ($B$ for bad) as:

$$\chi(i,k) = \begin{cases} G & \text{if } \eta^{\pm}_\xi(x, kT|\log \epsilon|) = 1, \quad \forall x \in \Delta_i(\ell_2); \\ B & \text{otherwise.} \end{cases}$$

Where the $\pm$ in $\eta^{\pm}_\xi(x,t)$ means that we begin, in the graphical construction, with the initial configurations $\mathbf{1}$ and $-\mathbf{1}$. Then calling:

$$p_\epsilon(k) = P^{(\epsilon)}_{\mathbf{1},-\mathbf{1}}(\chi(i,k) = B). \tag{16}$$

We have that:

**Proposition 2.1** *For $\epsilon$ small enough there exists $c(\epsilon) > 0$ and $a > 0$ such that, for all $k \geq 1$:*

$$p_\epsilon(k) \leq c(\epsilon)\epsilon^{ak}. \tag{17}$$

In Proposition 5.1 of [BPSV2], the authors prove that $P^{(\epsilon)}_{\mathbf{1},-\mathbf{1}}(\chi(i,k) = D) \leq c(\epsilon)\epsilon^{ak}$, where $\chi(i,k) = D$ means that there is a discrepancie at the $i$-cube at time $kT|\log \epsilon|$, we follow the proof and make the apropiate changes to our case. Using this proposition we can conclude Lemma 2.2. By (14):

$$\begin{aligned} \mu_\epsilon(\sigma(0)) &\geq -2P^{(\epsilon)}_{-\mathbf{1}}[|M_n(\sigma,t) - m_+| > \xi] - \xi + m_+ \\ &\geq -2p_\epsilon(k) - \xi + m_+ \\ &\geq -2(c(\epsilon)\epsilon^{ak}) - \xi + m_+. \end{aligned} \tag{18}$$

As $k$ goes to infinity we get:

$$\mu_\epsilon(\sigma(0)) \geq m_+ - \xi, \tag{19}$$

so the Lemma 2.2 is proved.

**Proof (Proposition 2.1)**
First we quote the Proposition 2.1 from [BPSV2]:

**Proposition 2.2** *For any $\xi > 0$ small enough there exists $T_0 > 0$ such that given $T \geq T_0$, $n \geq 1$ we can find constants $c_n > 0$ such that the following holds. Let $\xi > 0, \sigma$ such that $\eta_\xi(x, 0) = 1$ for all $x \in \Delta(\ell_2)$. Then:*

$$P_\sigma^{(\epsilon)}\left(\eta_\xi(x,t) = 1, \quad \textit{for all } x \in \Delta(3\ell_2)\right) \geq 1 - c_n \epsilon^n. \tag{20}$$

Next we fix $n \geq d+2$, $\xi$ small enough and $T$ as in the last proposition. We say that $(i, k)$ is connected with $(j, k+1)$, in sinmbols $(i, k) \to (j, k+1)$, if either $i = j$ or $\Delta_i(\ell_2)$ and $\Delta_j(\ell_2)$ are contiguous. This are the only (oriented) conections in $\mathbb{Z}^d \times \mathbb{N}$. We say that the site $(i, k)$ is bad if $\chi(i, k) = B$.

Given $(i, k), k \geq 2$, we call $\mathcal{C}_{(i,k)}$ the bad cluster connected (from below) to $(i, k)$ as the set:

$$\{(j, h) : 1 \leq h \leq k, j \in \mathbb{Z}^d \text{ there exists a path from } (i, k) \text{ to } (j, h) \text{ made of bad sites}\}. \tag{21}$$

Note that for all $(i, k) \in \mathcal{C}_{(i,k)}$ we have $\chi(i, k) = B$. Also if $\mathcal{C}_{(i,k)} = \emptyset$ for $k \geq 2$ then $\chi(j, k-1) = G$ for $j = i$ and for all $j$ such that $\Delta_i(\ell_2)$ and $\Delta_j(\ell_2)$ are contiguous.

If $\mathcal{C}_{(i,k)} \cap (\mathbb{Z}^d \times \{1\}) = \emptyset$, we set $\tau(i, k)$ the largest time $1 \leq h \leq k$ such that for all $j$, $(j, h) \notin \mathcal{C}_{(i,k)}$. In particular $\chi(j, h) = G$ for all $(j, h)$ connected to $\mathcal{C}_{(i,k)}$. If $\mathcal{C}_{(i,k)} \cap (\mathbb{Z}^d \times 1) \neq \emptyset$, set $\tau(i, k) = 0$.

We can decompose $\{\chi(i, k) = B\}$, for $k \geq 2$ as:

$$\bigcup_{1 \leq h \leq k} \bigcup_{C_{(i,k)}} \{\chi(i, k) = B, \tau(i, k) = h, \mathcal{C}_{(i,k)} = C_{(i,k)}\} \quad \bigcup \{\chi(i, k) = B, \tau(i, k) = 0\}. \tag{22}$$

Let us condition the process from 0 up to time $hT|\log \epsilon|$, and call $P_h^{(\epsilon)}$ this conditional law. Then:

$$p_\epsilon(k) \leq \sum_{1 \leq h \leq k} \sum_{C_{(i,k)}} P^{(\epsilon)} P_h^{(\epsilon)}(\{\chi(i, k) = B, \tau(i, k) = h, \mathcal{C}_{(i,k)} = C_{(i,k)}\}) + P^{(\epsilon)}(\tau(i, k) = 0). \tag{23}$$

If $\mathcal{C}_{(i,k)} = \emptyset$ then $\tau(i, k) = k - 1$ so that:

$$\{\chi(i, k) = B, \mathcal{C}_{(i,k)} = \emptyset\} \subseteq \{\chi(i, k-1) = G, \chi(i, k) = B\}. \tag{24}$$

We estimate the probability of the right hand side, by Proposition 2.2: for all $n$ there exists $c_n$ such that for all $\epsilon > 0$ small enough it holds:

$$P_{k-1}^{(\epsilon)}(\chi(i, k) = B, \mathcal{C}_{(i,k)} = \emptyset) \leq c\epsilon^n. \tag{25}$$

If $\mathcal{C}_{(i,k)} = C_{(i,k)} \neq \emptyset$, let us fix $\tau(i, k) = h$, $1 \leq h \leq k-2$ and define the set:

$$A = \{(j, \tilde{k}) : h \leq \tilde{k} \leq k-2, \chi(j, \tilde{k}) = G, \quad \exists \tilde{j} : (j, \tilde{k}) \longrightarrow (\tilde{j}, \tilde{k}+1) \in C_{(i,k)}\}. \tag{26}$$

Since $\tau(i, k) = h$ the cardinality of $A$ is at least $q = 2d(k-h)$.

To control the influence of the process outside of a ball we only need to control the "dual branching process": $(Y_{t,s}^x)_{s \leq t}$ associated with the process in the graphical construction (see Appendix), this is the content of the next Proposition, proved in [BPSV2]. For $r > 0$ call $B_r$ the closed ball centered at 0 with radius $r$ with the sup norm in $\mathbb{R}^d$, then:

**Proposition 2.3** *There exists positive constants $c_0$ and $c^*$ such that for all $\epsilon \le 1$ and all $t \ge |\log\epsilon|$ it holds:*

$$P^{(\epsilon)}(\{\exists \quad 0 \le s' < s'' \le t : Y^0_{s'',s'} \not\subseteq B_{c_0\epsilon^{-1}t}\}) \le c^* e^{-t}. \tag{27}$$

Now define:

$$S^i_{k,h,r} \equiv \{Y^{\Delta_i(\ell_2)}_{kT|\log\epsilon|,hT|\log\epsilon|} \not\subseteq \bigcup_{|j'-i|<r} \Delta_{j'}(\ell_2)\}. \tag{28}$$

and observe that this event belongs to the $\sigma-$field generated by the marks in the time interval $[hT|\log\epsilon|, kT|\log\epsilon|]$, which is conditionally independient of what happens before $hT|\log\epsilon|$. So that for $r_0 = c_0(T \wedge (d+2))$, and $c_0$ as in proposition 2.3, we get

$$P^{(\epsilon)}_h(S^i_{k,h,r}) \le \{|\Delta_i(\ell_2)|c^* e^{-(r-1)|\log\epsilon|/c_0}\mathbf{1}_{(r-1)\ge(k-h)r_0}\} + \mathbf{1}_{(r-1)<(k-h)r_0}. \tag{29}$$

We may take a subset $A'$ of $A$ whose cardinality is not smaller than $q/(4r_0)^d$ and such that any two elements $(j,k')$ and $(j',k')$ in $A'$ have $|j-j'| > 2r_0$. We condition the process until time $k'T|\log\epsilon|$, where $k'$ is the largest time for which $(j,k')$ is in $A'$ (for some $j$). Let us denote by $(j'_1,k')\ldots(j'_p,k')$ the points in $A'$ with the given $k'$ and by $(j_1,k'+1)\ldots(j_p,k'+1)$ the points in $C_{(i,k)}$ connected to the above ones in $A'$. Let $\ell = 1\ldots p$ and

$$V^{\ell,k',u} = \begin{cases} S^{j_\ell}_{k'+1,k',r_0+1}, & \text{if } u=1; \\ \{\chi(j'_\ell,k') = G, \chi(j_\ell,k'+1) = B\} \cap [S^{j_\ell}_{k'+1,k',r_0+1}]^c, & \text{if } u=-1. \end{cases}$$

We then have

$$P^{(\epsilon)}_{k'}(\bigcap_{\ell=1}^p \{\chi(j'_\ell,k') = G, \chi(j_\ell,k'+1) = B\}) \le \sum_{u_1,\ldots,u_p}\prod_{\ell=1}^p P^{(\epsilon)}_{k'}(V^{\ell,k',u_\ell}). \tag{30}$$

By Propositions 2.3 and 2.2 there is $c>0$ such that

$$P^{(\epsilon)}_{k'}(V^{\ell,k',u_\ell}) \le c\epsilon^{(T\vee(d+2))-d-1} \le c\epsilon^p, \tag{31}$$

and

$$P^{(\epsilon)}_{k'}(\bigcap_{\ell=1}^p \{\chi(j'_\ell,k') = G, \chi(j_\ell,k'+1) = B\}) \le [c\epsilon]^p \sum_{u_1,\ldots,u_p} 1 \le [2c\epsilon]^p. \tag{32}$$

Iterating, we get

$$P^{(\epsilon)}_h(\{\chi(i,k) = B, \tau(i,k) = h, \mathcal{C}_{(i,k)} = C_{(i,k)}\}) \le [2c\epsilon]^{q/(4r_0)^d}. \tag{33}$$

So that in (23) we get:

$$p_\epsilon(k) \le \sum_{h=1}^k c\epsilon^{\tilde{a}(k-h)} + P^{(\epsilon)}(\tau(i,k) = 0). \tag{34}$$

The proof is then reduced to show that there exists an $c > 0$ and $g_\epsilon > 0$ such that for all $k \geq 2$ it holds

$$P^{(\epsilon)}(\tau(i,k) = 0) \leq e^{-g_\epsilon k^{d+1}} + c\epsilon^{\tilde{a}k}, \tag{35}$$

and

$$p_\epsilon(1) \leq e^{-g_\epsilon}. \tag{36}$$

Let $k \geq 2$ and $A$ as before. We distinguish two cases $|A| \leq N_0$ and $|A| > N_0$, with $N_0 = 2d(k/2)$.

By (33):

$$P^{(\epsilon)}(\{\tau(i,k) = 0, |A| > N_0\}) \leq [2c\epsilon]^{N_0/(4r_0)^d}, \tag{37}$$

which is bounded as the second term of r.h.s. of (35).

By simple geometric considerations, there is $a > 0$ such that

$$\inf_{\tau(i,k)=0, |A| \leq N_0} |C_{(i,k)}| \geq ak^{d+1}. \tag{38}$$

So we the proof of (35) is reduced to prove that:

$$P^{(\epsilon)}(\{|A| \leq N_0, |C_{(i,k)}| \geq ak^{d+1}\}) \leq e^{-g_\epsilon k^{d+1}}. \tag{39}$$

If $|A| \leq N_0$ and $|C_{(i,k)}| \geq ak^{d+1}$, then there is $a' > 0$ so that there are at least $a'k^d$ connections $(j,k') \to (j,k'+1)$ in $C_{(i,k)}$. Let $\mathcal{E}$ be an element in the space of marks (see section 5) with the following properties:

**i)** There are no marks $\mathcal{N}^\epsilon_{x,e_i}$ with either $x$ or $x + e_i$ in the cube $\Delta(\ell_2)$ in the time interval $[0, T|\log\epsilon|]$.

**ii)** $\eta^\pm_\xi(x, T|\log\epsilon|) = 1$ for all $x$ in the cube $\Delta(\ell_2)$.

To find $P^{(\epsilon)}(\mathcal{E})$ note that:

$$\begin{aligned} &P^{(\epsilon)}(\{\forall x \in \Delta(\ell_2) \quad \forall i = 1,\ldots,d: \quad \mathcal{N}^\epsilon_{x,e_i} > T|\log\epsilon|\}) \\ &= \quad [P^{(\epsilon)}(\{\mathcal{N}^\epsilon_{0,e_i} > T|\log\epsilon|\})]^{2d|\Delta(\ell_2)|} \\ &= \quad [e^{-\epsilon^{-2}T|\log\epsilon|}]^{2d(\epsilon^{-1}|\log\epsilon|)^d} \\ &> \quad e^{-2dT\epsilon^{-5d}}. \end{aligned} \tag{40}$$

On the other hand, calling $\tau^x_n, \quad , x \in \mathbb{Z}^d, n \geq 0$, the times when the Poisson marks, with rates $c_{max} = (1+\gamma)^2 + \mu/2$, rings. Suppose that

$$\tau^x_1 < T|\log\epsilon| < \tau^x_2, \quad x \in \Delta(\ell_2). \tag{41}$$

Observe that in order to show that

$$c_\mu(x,\sigma) > U^{x,+}_1 c_{max}. \tag{42}$$

it suffices to show that

$$U^{x,+}_1 < \{(1-\gamma)^2 + \mu/2\}/c_{max}. \tag{43}$$

since $c_\mu(x,\sigma) \geq (1+\gamma^2)$. Conditions (41), (42) in turn implies (ii).

But

$$\begin{aligned} & P^{(\epsilon)}(\{\forall x \in \Delta(\ell_2) : \tau_1^x < T|\log\epsilon| < \tau_2^x, U_1^{x,+} < \{(1-\gamma)^2 + \mu/2\}/c_{max}\}) \\ &= \prod_{x\in\Delta(\ell_2)} P^{(\epsilon)}(\tau_1^x < T|\log\epsilon| < \tau_2^x) P^{(\epsilon)}(U_1^{x,+} < \{(1-\gamma)^2 + \mu/2\}/c_{max}) \\ &= \left[ c_{max} T|\log\epsilon| e^{-c_{max}T|\log\epsilon|} \left( \frac{(1-\gamma)^2 + \mu/2}{c_{max}} \right) \right]^{|\Delta(\ell_2)|}. \end{aligned} \tag{44}$$

By (40), (44) and the independence of the marks process:

$$P^{(\epsilon)}(\mathcal{E}) > \tilde{c} e^{-c\epsilon^{-a}}.$$

Now if we call $g_\epsilon = |\log(1 - P^{(\epsilon)}(\mathcal{E}))|$ then:

$$\begin{aligned} & P^{(\epsilon)}(\{|A| \leq N_0, |C_{(i,k)}| \geq ak^{d+1}\}) \\ &\leq P^{(\epsilon)}(\{\text{There exists} \quad a'k^d \text{ conections } (j,k') \to (j,k'+1) \text{ in } C_{(i,k)}\}) \\ &\leq [P^{(\epsilon)}(\mathcal{E}^c)]^{a'k^{d+1}} \\ &= e^{-a' g_\epsilon k^{d+1}}. \end{aligned} \tag{45}$$

Iterating, and taking $k \geq k(\epsilon) = (\frac{\tilde{a}|\log\epsilon|}{g_\epsilon})^{1/d}$, so that $g_\epsilon(k/2) \geq \tilde{a}|\log\epsilon|$, we get:

$$\begin{aligned} p_\epsilon(k) &\leq c \left( \sum_{h=0}^{k} \epsilon^{\tilde{a}(k-h)} e^{c(k-h)} e^{-g_\epsilon h^{d+1}} \right) \\ &\leq c \left( \epsilon^{\tilde{a}k/2} + e^{-g_\epsilon (k/2)^{d+1}} \right) \\ &\leq c\epsilon^{\tilde{a}k/2} (\epsilon^{-\tilde{a}k/2} e^{-g_\epsilon (k/2)^{d+1}}). \end{aligned} \tag{46}$$

Finally if we set $c(\epsilon) = C\epsilon^{-\tilde{a}k/2} e^{-g_\epsilon (k/2)^{d+1}}$ then

$$p_\epsilon(k) \leq c(\epsilon) \epsilon^{\tilde{a}k/2}.$$

So the proof is concluded.

Note that since $g_\epsilon > |\log(1 - \tilde{c}e^{-c\epsilon^{-a}})|$ then:

$$\begin{aligned} c(\epsilon) &\leq \epsilon^{-\frac{\tilde{a}}{2}(\frac{\tilde{a}|\log\epsilon|}{g_\epsilon})^{1/d}} e^{-g_\epsilon (k/2)^{d+1}} \\ &\leq (e^{-g_\epsilon^{1/d}})^{\frac{1}{2}(\tilde{a}|\log\epsilon|)^{1+1/d}} \\ &\leq \exp\left( \frac{1}{\phantom{2}} \right. \end{aligned}$$

## 2.2 Proof of Theorem 2.3

We will use a result of Liggett (Theorem 4.20 in [L1]). Before stating this result we shall modify the expression of the generator:

$$L_{\mu}^{(\epsilon)} = \epsilon^{-2} L_0 + L_{\mu}, \tag{48}$$

in a way that highlights its local dynamics. Let us consider a finite subset $T$ of $\mathbb{Z}^d$, and for each $\sigma \in \Omega$, let us define the rate, $c_T(\sigma, d\xi), \xi \in \{-1,1\}^T$ at which a transition occurs involving only the coordinates on $T$.

We begin with the stirring process. Note that if $T = \{x, y\}, |x - y| = 1$ and if $\xi(x) = \sigma(y)$ and $\xi(y) = \sigma(x)$ then $c_T(\sigma, \xi) = \epsilon^{-2}$. So that $\sigma^{x,y} \equiv \sigma^{\xi}$, where $\sigma^{\xi}$ is the configuration that is equal to $\sigma$ in the complement of $T$, and equal to $\xi$ on $T$.

Next, in the Glauber case, if $T = \{x\}$ then $c_T(\sigma, \xi) = c_{\mu}(x, \sigma)$, when $\xi \in \{-1,1\}^{\{x\}}$ is such that $\xi(x) = -\sigma(x)$. And let $c_T(\sigma, d\xi) = 0$ in the other cases.

In this way, we can rewrite the generator as:

$$L_{\mu} f(\sigma) = \sum_T \int_{\{-1,1\}^T} c_T(\sigma, d\xi)[f(\sigma^{\xi}) - f(\sigma)]. \tag{49}$$

Now call $C_T$ the maximun transition rate inolving the coordinates in $T$:

$$C_T \equiv \sup\{c_T(\sigma, \{-1,1\}^T) : \sigma \in \Omega\},$$

and note that, if $|T| \geq 2$ or if $T = \{x, y\}$ with $|x - y| > 1$ then $C_T = 0$. Also if we denote $c_{max} \equiv (1 + \gamma)^2 + \mu/2$, then

$$C_T \leq \epsilon^{-2} \vee c_{max} \leq 2\epsilon^{-2}$$

for $\epsilon > 0$ small enough.

For $u \in \mathbb{Z}^d$ and $T$ as before, define:

$$c_T(u) = \sup\{\|c_T(\sigma_1, d\xi) - c_T(\sigma_2, d\xi)\|_T : \sigma_1(y) = \sigma_2(y) \text{ for all } y \neq u\}, \tag{50}$$

where $\|\cdot\|_T$ refers to the total variation norm in $\{-1,1\}^T$. This measures the dependence of the rate $c_T(\sigma, d\xi)$ in the coordinate $u$. If $T = \{u\}$ and $f$ is a function in $\{-1,1\}^T$, $|f| \leq 1$ and $\sigma_1, \sigma_2 \in \Omega$ are such that $\sigma_1(u) \neq \sigma_2(u)$ then:

$$\begin{aligned} & \left|\int_{\{-1,1\}^T} f(\xi) c_T(\sigma_1, d\xi) - \int_{\{-1,1\}^T} f(\xi) c_T(\sigma_2, d\xi)\right| \\ \leq & \left|\frac{1}{d} \sum_{i=1}^{d} [\gamma(\sigma_1(u + e_i) + \sigma_1(u - e_i)) + \mu/2](f(1) + f(-1))\right| \\ \leq & 4\gamma + \mu, \end{aligned}$$

so that $c_T(u) \leq 4\gamma + \mu$.

Moreover, if $T$ has two elements, then we have two cases. First, if $T = \{x, y\}$ with $|x - y| = 1$ , and $x \neq u \neq y$ then $c_T(u) = 0$; and second if $T = \{x, u\}$, $|x - u| = 1$, then similary as before one gets that $c_T(u) \leq \epsilon^{-2}$.

And for $T$ in the other cases we have $c_T(u) = 0$.

Now, for all $x \neq u$ define:

$$\gamma(x, u) = \sum_{T \ni x} c_T(u). \tag{51}$$

Then by translation invariance

$$\gamma(x, u) = \gamma(0, x - u) = c_{\{x,u\}}(u) \leq \epsilon^{-2},$$

when $|x - u| = 1$, and $\gamma(x, u) = 0$ if $|x - u| > 2$.

We say that the process has finite range $N$ if two conditions hold:

a) $C_T = 0$ for $|T| \geq N$, and b) $\gamma(x, u) = 0$ if $|x - u| \geq N$.

In our process, $|T| \geq 2$ or $T = \{x, y\}$ with $|x-y| > 1$ implies $C_T = 0$. Also $\gamma(x, u) = 0$ if $|x - u| > 2$. So that the range of our process is 2.

Finally let us define

$$M = \sup_{x \in \mathbb{Z}^d} \sum_u \gamma(x, u),$$

in particular $M \leq \epsilon^{-2}$ in our process.

Now we apply Theorem 4.18 of Chapter I in [L1], that we state below. We denote $S(t)$ the semigroup of the process :

**Theorem 2.4** *If the process is of finite range then there exists $\delta > 0$ and $K > 0$, such that*

$$\|S(t)(f(\sigma)g(\sigma)) - S(t)(f(\sigma))S(t)(g(\sigma))\| \leq K[|||f||| \; |||g|||] \exp(4Mt - \delta d(R_1, R_2)), \tag{52}$$

where $\|h\| = \sup_{\sigma \in \Omega} |h(\sigma)|$, $f$ and $g$ are local functions; $R_1 = supp(f), R_2 = supp(g)$, so that $d(R_1, R_2)$ is the distance between $R_1, R_2$; $|||f||| = \sum_x \Delta_f(x)$ and

$$\Delta_f(x) = \sup\{|f(\eta) - f(\xi)| : \eta, \xi \in \Omega, \eta(y) = \xi(y) \text{ for all } y \neq x\}.$$

We sketch briefly the proof.

*Proof*

We have that:

$$\begin{aligned} &\|S(t)(f(\sigma)g(\sigma)) - S(t)(f(\sigma))S(t)(g(\sigma))\| \\ &\leq \sum_{x,y} \Delta_f(x)\Delta_g(y) \sum_{u,v} [\sum_{T \ni u,v} C_T] \int_0^t \gamma_s(u - x)\gamma_s(v - y)ds, \end{aligned} \tag{53}$$

where $\gamma_s(\cdot)$ is the kernel of the semigoup generated by $\Gamma\beta(u) = \sum_x \beta(x)\gamma(x - u)$ and $\beta \in \ell_1(\mathbb{Z})$, that is $e^{s\Gamma}\beta(u) = \sum_x \beta(x)\gamma_s(x - u)$. The main part proof is to bound :

$$\sup_s e^{-4Ms} \sum_{u,v,y} e^{\delta|y|} \left( \sum_{T \ni u,v} C_T \right) \gamma_s(u)\gamma_s(v - y). \tag{54}$$

First take $\delta > 0$ such that

$$\sum_u \gamma(0,u)e^{\delta|u|} \leq 2M. \tag{55}$$

Since $M = \sum_u \gamma(0,u)$ is the norm of the operator $\Gamma$ then

$$\sum_u \gamma_s(u)e^{\delta|u|} \leq e^{2Ms},$$

so that:

$$\begin{aligned}
&\sup_s e^{-4Ms} \sum_{u,v,y} e^{\delta|y|}[\sum_{T\ni u,v} C_T]\gamma_s(u)\gamma_s(v-y) \\
&\leq \sup_s e^{-4Ms} \sum_{u,v,y}[\sum_{T\ni u,v} C_T]\gamma_s(u)e^{\delta|u|}\gamma_s(v-y)e^{\delta|v-y|}e^{\delta|v-u|} \\
&\leq (\sup_{u,v}[\sum_{T\ni u,v} C_T])(e^{\delta N}|\{v : |u-v| = 1, |u| = 1; |v-y| = 1; u,v,y \in Z^d\}|) \\
&= K,
\end{aligned} \tag{56}$$

where $N$ is the range of the process, and $|A|$ is the cardinality of the set $A$.

Now, back to our process. If we denote $f_x(\eta) = \eta(x)$ then $supp(f_x) = \{x\}$, and $\|\!|f_x|\!\| = 2$. In general, if $f_{x_1,\ldots,x_n}(\eta) = \prod_{i=1}^n \eta(x_i)$ then $\|\!|f_{x_1,\ldots,x_n}|\!\| = 2^n$. In (55), on the last proof is enough to take $\delta = \log 2$. Our process has range two and $M \leq \epsilon^{-2}$, so that in (56) we have $K \leq C\epsilon^{-2}$. Apliying this Theorem to the local functions $f_0$ and $f_x$, we conclude that:

$$\|\mathbb{E}^{(\epsilon)}(\sigma_t(0)\sigma_t(x)) - \mathbb{E}^{(\epsilon)}(\sigma_t(0))\mathbb{E}^{(\epsilon)}(\sigma_t(x))\| \leq C\epsilon^{-2}\exp(8\epsilon^{-2}t - (\log 2)|x|). \tag{57}$$

Now we use the exponential convergence to the ergodic measure to conclude the proof. Recall that for $\epsilon$ small enough there are $c(\epsilon) = ce^{ce^{\epsilon^{-a}}}$, see (47), so that:

$$\mathbb{E}^{(\epsilon)}_{\mathbf{1},-\mathbf{1}}\left(\sigma^+(0,t) - \sigma^-(0,t)\right) \leq c(\epsilon)e^{-\lambda t}. \tag{58}$$

Then we obtain

$$\begin{aligned}
|\mathbb{E}^{(\epsilon)}_\sigma(\sigma(0,t)) - \mathbb{E}^{(\epsilon)}_{\mu_\epsilon}(\sigma(0))| &\leq \mathbb{E}^{(\epsilon)}_{\mathbf{1}}(\sigma(0,t)) - \mathbb{E}^{(\epsilon)}_{-\mathbf{1}}(\sigma(0,t)) \\
&\leq \mathbb{E}^{(\epsilon)}_{\mathbf{1},-\mathbf{1}}(\sigma^+(0,t) - \sigma^-(0,t)) \\
&\leq c(\epsilon)e^{-\lambda t}.
\end{aligned} \tag{59}$$

Moreover

$$\begin{aligned}
|\mathbb{E}^{(\epsilon)}_\sigma(\sigma(x,t)\sigma(y,t)) &- \mathbb{E}^{(\epsilon)}_{\mu_\epsilon}(\sigma(x)\sigma(y))| \leq |\mathbb{E}^{(\epsilon)}_{\sigma,\mu_\epsilon}(\sigma(x,t)\sigma(y,t)) - (\sigma^\epsilon(x,t)\sigma^\epsilon(y,t))| \\
&\leq \mathbb{E}^{(\epsilon)}_{\mathbf{1},\mu_\epsilon}[(\sigma^+(x,t) - \sigma^\epsilon(x,t)) + (\sigma^+(y,t) - \sigma^\epsilon(y,t))] \\
&\leq 2\mathbb{E}^{(\epsilon)}_{\mathbf{1},-\mathbf{1}}(\sigma^+(0,t) - \sigma^-(0,t)) \\
&\leq 2c(\epsilon)e^{-\lambda t}.
\end{aligned} \tag{60}$$

Let $0 < s < t$, then, after using the previous bounds:

$$\begin{aligned}
&\|\mathbb{E}^{(\epsilon)}(\sigma_t(0)\sigma_t(x)) - \mathbb{E}^{(\epsilon)}(\sigma_t(0))\mathbb{E}^{(\epsilon)}(\sigma_t(x))\| \\
\leq\ & \|\mathbb{E}^{(\epsilon)}(\sigma_s(0)\sigma_s(x)) - \mathbb{E}^{(\epsilon)}(\sigma_s(0))\mathbb{E}^{(\epsilon)}(\sigma_s(x))\| \\
+\ & \|\mathbb{E}^{(\epsilon)}(\sigma_t(0)\sigma_t(x)) - \mathbb{E}^{(\epsilon)}(\sigma_s(0)\sigma_s(x))\| \\
+\ & \|\mathbb{E}^{(\epsilon)}(\sigma_t(0))\|\|\mathbb{E}^{(\epsilon)}(\sigma_t(x)) - \mathbb{E}^{(\epsilon)}(\sigma_s(x))\| \\
+\ & \|\mathbb{E}^{(\epsilon)}(\sigma_t(x))\|\|\mathbb{E}^{(\epsilon)}(\sigma_t(0)) - \mathbb{E}^{(\epsilon)}(\sigma_s(0))\| \\
\leq\ & 4C\epsilon^{-2}\exp(8\epsilon^{-2}s - \log 2|x|) + (2c(\epsilon)e^{-\lambda t} + 2c(\epsilon)e^{-\lambda s}) + 2(c(\epsilon)e^{-\lambda t} + c(\epsilon)e^{-\lambda s}) \\
\leq\ & 4C\epsilon^{-2}\exp(8\epsilon^{-2}s - \log 2|x|) + 8c(\epsilon)e^{-\lambda s}. \qquad (61)
\end{aligned}$$

Now, if we take $\tilde{\delta} = \min\{\log 2, \lambda\}$ and $s = \frac{\tilde{\delta}|x|}{\tilde{\delta}+8\epsilon^{-2}}$, then we have $8\epsilon^{-2}s - \log 2|x| \leq -\frac{\tilde{\delta}^2|x|}{\tilde{\delta}+8\epsilon^{-2}}$, so that

$$\begin{aligned}
|\mathbb{E}^{(\epsilon)}_{\sigma}(\sigma(0,t)\sigma(x,t)) - \mathbb{E}^{(\epsilon)}_{\sigma}(\sigma(0,t))\mathbb{E}^{(\epsilon)}_{\sigma}(\sigma(x,t)) &\leq \max\{16\epsilon^{-2}, 8c(\epsilon)\}e^{-\frac{\tilde{\delta}^2}{\tilde{\delta}+8\epsilon^{-2}}|x|} \\
&\leq ce^{-c'\epsilon^{-\tilde{a}}}e^{-c\epsilon^2|x|}, \qquad (62)
\end{aligned}$$

for $t > s$.

To prove the theorem let $t$ goes to infinity.

# 3 Appendix: Graphical Representation

For each $x$ let $\mathcal{N}_x^+$, $\mathcal{N}_x^-$ be a Poisson process of intensity $c_{max} = (1+\gamma)^2 + \mu/2$, and for each bond $(x, x+e_i)$ let $\mathcal{N}^{\epsilon}_{x,x+e_i}$ be a Poisson process of intensity $\epsilon^{-2}$. Let also $U_n^{x,+}$, $U_n^{x,-}$, $n \geq 1$ be i.i.d. random variables uniformily distribuited in $(0,1)$, $x \in \mathbb{Z}^d$, all this variables are independient of each other. The graphical representation is a realization of the process for any initial configuration. If $\mathcal{N}^{\epsilon}_{x,x+e_i}$ apears we exchange the spins at $x, x+e_i$, and at the time of the $n$-th mark of $\mathcal{N}_x^-$ (resp.$\mathcal{N}_x^+$) we flip the spin at $x$ if and only if $\sigma(x) = 1$ and $c_\mu(x,\sigma) > U_n^{x,+}$ (resp. $\sigma(x) = -1$ and $c_\mu(x,\sigma) > U_n^{x,-}$). This rule defines a process equivalent to the Glauber+ Kawasaki, see [BPSV2].

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