Lifetime Data Anal (2010) 16:409–430
DOI 10.1007/s10985-010-9161-1

The beta modified Weibull distribution

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Received: 24 October 2008 / Accepted: 27 February 2010 / Published online: 18 March 2010

Abstract A five-parameter distribution so-called the beta modified Weibull distribution is defined and studied. The new distribution contains, as special submodels, several important distributions discussed in the literature, such as the generalized modified Weibull, beta Weibull, exponentiated Weibull, beta exponential, modified Weibull and Weibull distributions, among others. The new distribution can be used effectively in the analysis of survival data since it accommodates monotone, unimodal and bathtub-shaped hazard functions. We derive the moments and examine the order statistics and their moments. We propose the method of maximum likelihood for estimating the model parameters and obtain the observed information matrix. A real data set is used to illustrate the importance and flexibility of the new distribution.

Keywords Beta distribution · Exponentiated exponential · Exponentiated Weibull · Generalized modified Weibull · Maximum likelihood · Modified Weibull · Observed information matrix · Weibull distribution

1 Introduction

The Weibull distribution, having exponential and Rayleigh as special cases, is a very popular distribution for modeling lifetime data and for modeling phenomenon with

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monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modeling phenomenon with non-monotone failure rates such as the bathtub shaped and the unimodal failure rates which are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. An example of bathtub shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low level. It then remains at that level for quite a few years before picking up again. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

According to Nelson (1990, p. 27), the distributions which permit a bathtub fit are sufficiently complex. On the other hand more flexible distributions usually require five or more parameters. However, more recently, the generalized gamma (GG) and generalized F (GF) distributions, both under other parametrizations, were used in applications of survival analysis, see Cox et al. (2007) and Cox (2008), respectively. The GF distribution was used for the logarithm of the failure time.

Alternatively, other works had introduced new distributions for modeling bathtub shaped failure rate. For example, Rajarshi and Rajarshi (1988) presented a revision of these distributions and Haupt and Schabe (1992) considered a lifetime model with bathtub failure rates. But, these models do not present much practicability to be used. However, in the last few years, new classes of distributions were proposed based on modifications of the Weibull distribution to cope with bathtub shaped failure rate. A good review of some of these models is presented in Pham and Lai (2007). Between these, the exponentiated Weibull (EW) distribution introduced by Mudholkar et al. (1995, 1996), the additive Weibull distribution presented by Xie and Lai (1995), the extended Weibull distribution (Xie et al. 2002), the modified Weibull (MW) distribution proposed by Lai et al. (2003), the beta exponential (BE) distribution introduced by Nadarajah and Kotz (2006), the extended flexible Weibull distribution defined by Bebbington et al. (2007), the beta Weibull (BW) distribution studied by Lee et al. (2007) and the generalized modified Weibull (GMW) proposed by Carrasco et al. (2008).

In this work we introduce a new five-parameter distribution, so-called the beta modified Weibull (BMW) distribution, which contains several submodels such as the EW, exponentiated exponential (EE) (Gupta and Kundu 1999, 2001), MW, generalized Rayleigh (GR) (Kundu and Rakab 2005) and the GMW distribution, among several others. We feel that this generalization will attract wider applications in reliability and biology as well as in other areas of research. The new distribution due to its flexibility in accommodating all the forms of the risk function seems to be an important distribution that can be used in a variety of problems in modeling survival data. The BMW distribution is not only convenient for modeling comfortable bathtub-shaped failure rates but it is also suitable for testing goodness of fit of some special submodels such as the EW, MW and GMW distributions.

The rest of the paper is organized as follows. In Sect. 2, we define the BMW distribution and present some special cases. Section 3 provides expansions for its cumulative distribution function (cdf) and probability density function (pdf). We also show that the density function of the BMW distribution can be expressed as a mixture of MW density functions. The algebraic form for its reliability is obtained in Sect. 4. We derive in Sect. 5 general expansions for the moments of the new distribution. Section 6 is devoted to order statistics and their moments. We also present expansions for the L-moments (Hosking 1986) which are expectations of certain linear combinations of order statistics. They form the basis of a general theory which covers the summarization and description of theoretical probability distributions. In Sect. 7, we discuss maximum likelihood estimation and calculate the elements of the observed information matrix. The importance and flexibility of the BMW distribution is further emphasized in Sect. 8 by using it to model Aarset (1987) data. Section 9 ends with some conclusions.

2 The model definition

The idea of the BMW distribution stems from the following general class: if G denotes the cdf of a random variable then a generalized class of distributions can be defined by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_{0}^{G(x)} w^{a-1}(1-w)^{b-1}\mathrm{d}w \tag{1}$$

for $a > 0$ and $b > 0$, where $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio and $B_y(a, b) = \int_0^y w^{a-1}(1-w)^{b-1}\mathrm{d}w$ is the incomplete beta function. This class of generalized distributions has been receiving considerable attention over the last years, in particular after the recent works of Eugene et al. (2002) and Jones (2004). Eugene et al. (2002) introduced what is known as the beta normal (BN) distribution by taking $G(x)$ in (1) to be the cdf of the normal distribution and derived some of its first moments. More general expressions for these moments were derived by Gupta and Nadarajah (2004). Nadarajah and Kotz (2004) introduced the beta Gumbel (BGu) distribution by taking $G(x)$ to be the cdf of the Gumbel distribution and provided closed form expressions for the moments, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation procedure. Nadarajah and Gupta (2004) introduced the beta Fréchet (BFr) distribution by taking $G(x)$ to be the Fréchet distribution, derived the analytical shapes of the density and hazard rate functions and calculated the asymptotic distribution of the extreme order statistics. Also, Nadarajah and Kotz (2006) worked with the BE distribution and obtained the moment generating function, the first four cumulants, the asymptotic distribution of the extreme order statistics and discussed the maximum likelihood estimation. Another distribution that happens to belong to Eq. 1 is the beta logistic distribution, which has been around for over 20 years (Brown et al. 2002), even if it did not originate directly from this equation.

We are motivated to introduce the BMW distribution because of the above generalizations, the wide usage of the Weibull distribution and the fact that the current generalization provides means of its continuous extension to still more complex situations. Lai et al. (2003) introduced the MW distribution having three parameters $\alpha > 0$, $\gamma > 0$ and $\lambda \geq 0$ with cdf and pdf given by

$$G_{\alpha,\gamma,\lambda}(x) = 1 - \exp\{-\alpha x^{\gamma} \exp(\lambda x)\} \tag{2}$$

and

$$g_{\alpha,\gamma,\lambda}(x) = \alpha x^{\gamma-1}(\gamma + \lambda x) \exp\{\lambda x - \alpha x^{\gamma} \exp(\lambda x)\}, \quad x > 0, \tag{3}$$

respectively. The parameters α and γ control the scale and shape of the distribution, respectively. The parameter λ is a kind of accelerating factor in the imperfection time and it works as a factor of fragility in the survival of the individual when the time increases. The Weibull (W) distribution is a special case of (3) when $\lambda = 0$. If, in addition to $\lambda = 0$, $\gamma = 1$ and $\gamma = 2$, we obtain the exponential (E) and Rayleigh (R) distributions, respectively.

The density corresponding to (1) can be written in the form

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1}\{1 - G(x)\}^{b-1} g(x), \tag{4}$$

where $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ is the density of the baseline distribution. The pdf $f(x)$ will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions as is the case of the MW distribution. Except for some special choices for $G(x)$ in (1), it would appear that the formula (4) will be difficult to deal with in generality.

We now introduce the BMW distribution by taking $G(x)$ in (1) to be the cdf (2) of the MW distribution. Hence, the general form for the BMW cdf is

$$F(x) = \frac{1}{B(a,b)} \int_{0}^{1-\exp\{-\alpha x^{\gamma} \exp(\lambda x)\}} \omega^{a-1}(1-\omega)^{b-1} \mathrm{d}\omega. \tag{5}$$

The BMW density function (for $x > 0$) can be written from (2) and (4) as

$$f(x) = \frac{\alpha x^{\gamma-1}(\gamma + \lambda x)\exp(\lambda x)}{B(a,b)} [1 - \exp\{-\alpha x^{\gamma} \exp(\lambda x)\}]^{a-1} \exp\{-b\alpha x^{\gamma} \exp(\lambda x)\}. \tag{6}$$

The BMW distribution contains as special cases several well-known distributions. For example, it simplifies to the BW distribution when $\lambda = 0$. If $\gamma = 1$, in addition to $\lambda = 0$, it reduces to the BE distribution. The GMW distribution is also a special case when $b = 1$. If $a = 1$ in addition to $b = 1$, it gives as special case the MW distribution. For $b = 1$ and $\lambda = 0$, the BMW distribution reduces to the EW distribution. If $\gamma = 1$ in addition to $b = 1$ and $\lambda = 0$, the BMW distribution becomes the EE distribution. For $\gamma = 2$, $\lambda = 0$ and $b = 1$, the BMW distribution reduces to

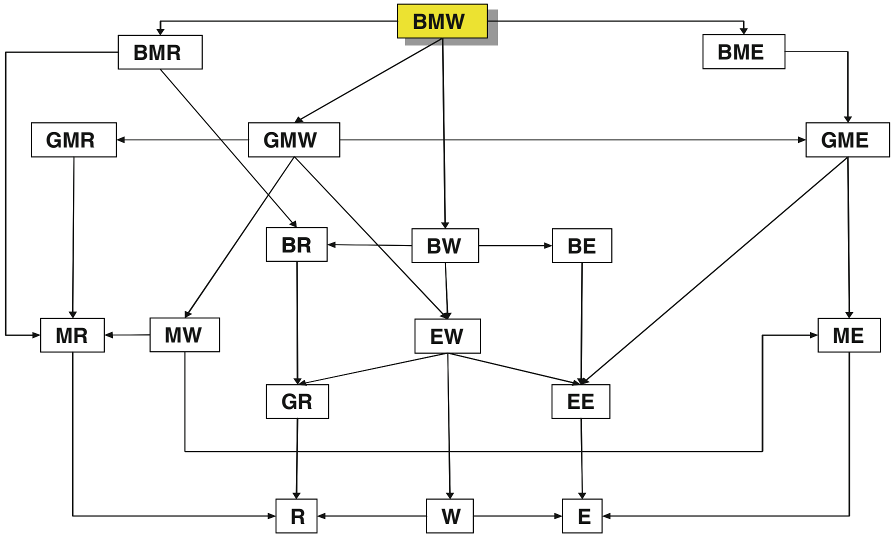

Fig. 1 Relationships of the BMW submodels

the GR distribution. The Weibull distribution is clearly a special case for $a = b = 1$ and $\lambda = 0$. Seventeen distributions included as special cases of the BMW distribution are displayed in Fig. 1, where the well-known submodels not defined before are: the beta modified Rayleigh (BMR), beta modified exponential (BME), generalized modified Rayleigh (GMR), generalized modified exponential (GME), beta Rayleigh (BR), modified Rayleigh (MR) and modified exponential (ME) distributions.

If X is a random variable with density (6), we write $X \sim \mathrm{BMW}(a, b, \alpha, \gamma, \lambda)$. The BMW distribution is easily simulated from (5) as follows: if V has a beta $B(a, b)$ distribution, then the solution of the nonlinear equation $X^{\gamma} \exp(\lambda X) = -\alpha^{-1} \log(1-V)$ has the $\mathrm{BMW}(a, b, \alpha, \gamma, \lambda)$ distribution. To simulate data from this nonlinear equation, we can use the matrix programming language Ox through *SolveNLE* subroutine (see Doornik 2007). The plots comparing the exact BMW densities and histograms from two simulated data sets for some parameter values are given in Fig. 2. These plots show that the simulated values are consistent with the BMW distribution.

The hazard function of the BMW distribution depends on the incomplete beta function ratio given by (for $x > 0$)

$$h(x) = \frac{\alpha x^{\gamma-1}(\gamma + \lambda x)\exp(\lambda x)}{B(a, b)[1 - I_{1-\exp\{-\alpha x^{\gamma}\exp(\lambda x)\}}(a, b)]}[1 - \exp\{-\alpha x^{\gamma}\exp(\lambda x)\}]^{a-1} \times \exp\{-b\alpha x^{\gamma}\exp(\lambda x)\}. \tag{7}$$

Some of the possible shapes of the density (6) and hazard function (7) for selected parameter values, including some well-known distributions, are illustrated in Figs. 3 and 4, respectively. A characteristic of the BMW distribution is that its hazard function can be bathtub shaped, monotonically increasing or decreasing and upside-down bathtub depending basically on the parameter values.

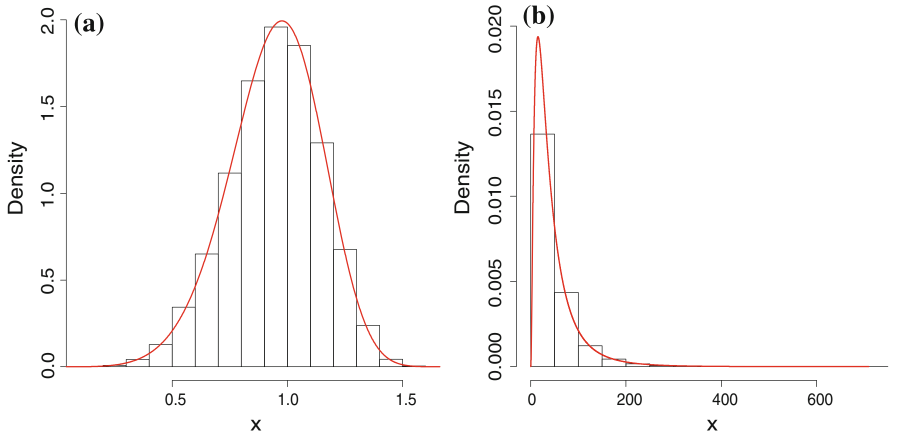

Fig. 2 Plots of the BMW densities for simulated data sets: **a** $a = 1.5, b = 0.5, \alpha = 0.5, \lambda = 1.5, \gamma = 3.0$ and **b** $a = 8, b = 1.0, \alpha = 0.43, \lambda = 0, \gamma = 0.5$

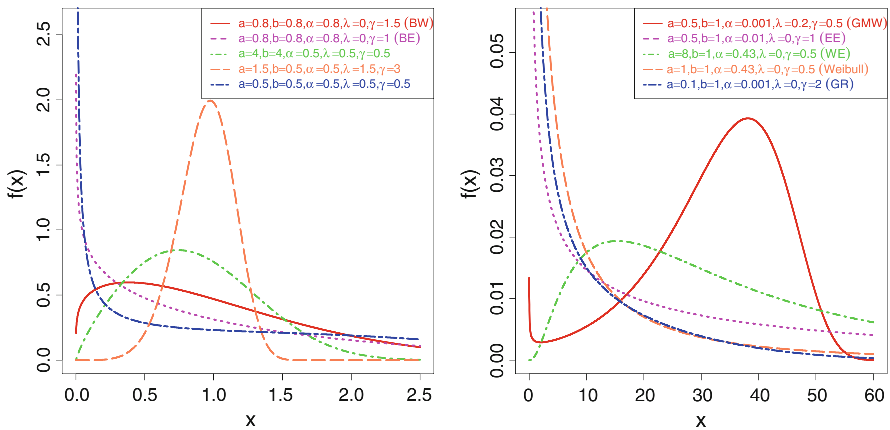

Fig. 3 Plots of the BMW density for some parameter values

3 Expansions for the distribution and density functions

We provide simple expansions for the BMW cdf depending on whether the parameter b (or a) is real non-integer or integer. The density in (6) is straightforward to compute using any statistical software. However, we show that the BMW density can be expressed as a mixture of MW distributions. This result is important to obtain some mathematical properties of the BMW distribution directly from those properties of the MW distribution.

We now give an expansion for $F(x)$ in terms of an infinite sum of MW cdf's. For $a > 0$ real non-integer, using the power series representation, yields

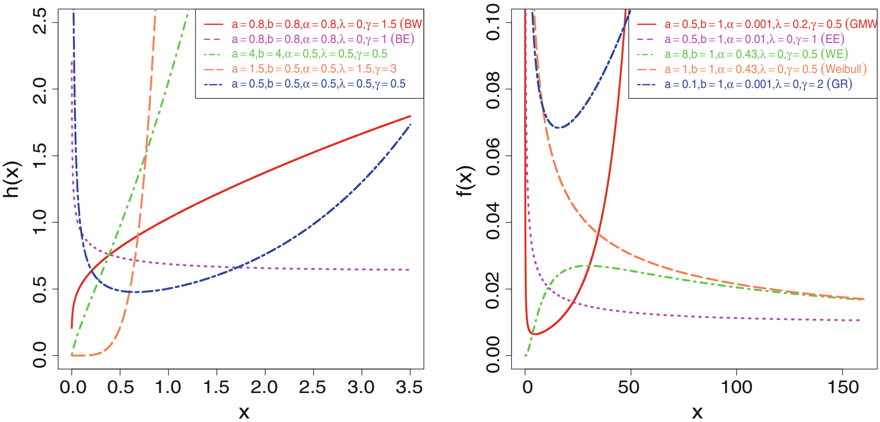

Fig. 4 Plots of the BMW hazard function for some parameter values

$$
\begin{aligned}
\int_0^x w^{a-1}(1-w)^{b-1}\mathrm{d}w &= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{\Gamma(a-j)j!} \int_0^x (1-w)^{b+j-1}\mathrm{d}w \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{\Gamma(a-j)(b+j)j!} \left[1-(1-x)^{b+j}\right],
\end{aligned}
$$

where $\Gamma(.)$ is the gamma function.

Hence,

$$
F(x) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(a)}{\Gamma(a-j)(b+j)j!} \{1-[1-G_{\alpha,\gamma,\lambda}(x)]^{b+j}\}
$$

and then

$$
F(x) = \sum_{j=0}^{\infty} w_j G_{\alpha(b+j),\gamma,\lambda}(x), \tag{8}
$$

where

$$
w_j = w_j(a,b) = \frac{(-1)^j \Gamma(a)}{B(a,b)\Gamma(a-j)(b+j)j!}
$$

are constants such that $\sum_{j=0}^{\infty} w_j = 1$ and $G_{\alpha(b+j),\gamma,\lambda}(x)$ is the cdf of the MW distribution with scale parameter $\alpha(b+j)$, shape parameter γ and accelerated parameter λ. If a is an integer, the index j in the previous sum stops at $a-1$. Expansion (8) is used throughout the paper for any a.

From the fact $\sum_{j=0}^{\infty} w_j = 1$, the BMW survival function has the following expansion

$$S(x) = 1 - F(x) = \sum_{j=0}^{\infty} w_j S_{\alpha(b+j),\gamma,\lambda}(x), \tag{9}$$

where $S_{\alpha(b+j),\gamma,\lambda}(x) = \exp\{-\alpha(b+j)x^{\gamma} \exp(\lambda x)\}$ is the survival function of the MW distribution with parameters $\alpha(b+j)$, γ and λ. The density of the BMW distribution follows immediately as

$$f(x) = \sum_{j=0}^{\infty} w_j g_{\alpha(b+j),\gamma,\lambda}(x). \tag{10}$$

Hence, the ordinary, central, factorial and inverse moments and the moment generating function of the BMW distribution could in principle follow from the same weighted infinite (or finite if a is an integer) linear combination of the corresponding quantities for MW distributions.

The MW density thus represents a particular case of (10) when $a = b = 1$. The GMW density is also a special case of (10) when only $b = 1$. Besides, while the transformation (1) is not analytically tractable in the general case, the formulae related to the BMW distribution turn out manageable (as it is shown in the rest of this paper), and with the use of modern computer resources with analytical and numerical capabilities, they may turn into adequate tools comprising the arsenal of applied statisticians.

Equations (8) and (10) represent the main results of this section. The advantage of the last expression is that it can be used to determine moments of the new distribution without any restrictions or conditions on its five parameters.

4 Reliability

In the area of stress-strength models there has been a large amount of work as regards estimation of the reliability $R = Pr(X_2 < X_1)$ when X_1 and X_2 are independent random variables belonging to the same univariate distribution. The algebraic form for R has been worked out for the majority of the well-known distributions. However, there are still many other distributions (including generalizations of the well-known distributions) for which the form of R has not been derived. Here, we obtain the form for the reliability R when X_1 and X_2 are independent random variables having the same BMW distribution.

The form of R can be expressed as

$$R = \int_0^{\infty} f(x)F(x)\mathrm{d}x. \tag{11}$$

Substituting the Eqs. 6 and 8 into (11), we obtain

$$R = \frac{\alpha}{B(a,b)} \sum_{j=0}^{\infty} w_j(a,b) \int_0^{\infty} x^{\gamma-1}(\gamma + \lambda x)e^{\lambda x}(1 - u^{b+j})(1 - u)^{a-1}u^b \mathrm{d}x,$$

where $u = \exp(-\alpha x^{\gamma} e^{\lambda x})$ and $w_j(a, b)$ is defined in Sect. 3. Hence, $\mathrm{d}u = u \log(u)[(\gamma + \lambda x)/x]\mathrm{d}x$ and R takes the form

$$R = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} w_j(a,b) \int_0^1 (1 - u^{b+j})(1-u)^{a-1} u^{b-1} \mathrm{d}x.$$

The last integral computed using Maple yields

$$R = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} w_j(a,b) \left[B(a,b) - \frac{\Gamma(a)\Gamma(2b+j)}{\Gamma(2b+j+a)} \right].$$

5 General formulae for the moments

We hardly need to emphasize the necessity and importance of moments in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). We now derive an infinite sum representation for the rth moment about zero of the BMW distribution, say μ'_r. From (10) we can obtain an elementary expression

$$\mu'_r = \sum_{j=0}^{\infty} w_j \tau_r(j), \tag{12}$$

where $\tau_r(j) = \int_0^{\infty} x^r g_{\alpha(b+j),\gamma,\lambda}(x)\mathrm{d}x$ denotes the rth moment of the MW distribution with parameters $\alpha(b + j)$, γ and λ.

Carrasco et al. (2008, Sect. 4) obtained an infinite representation for the rth moment of the MW distribution with the parameters above which can be written as

$$\tau_r(j) = \sum_{i_1,\ldots,i_r=1}^{\infty} \frac{A_{i_1,\ldots,i_r} \Gamma(s_r/\gamma + 1)}{[\alpha(b+j)]^{s_r/\gamma}}, \tag{13}$$

where

$$A_{i_1,\ldots,i_r} = a_{i_1} \ldots a_{i_r} \text{ and } s_r = i_1 + \cdots + i_r,$$

and

$$a_i = \frac{(-1)^{i+1} i^{i-2}}{(i-1)!} \left(\frac{\lambda}{\gamma} \right)^{i-1}.$$

Hence, the moments of the BMW distribution can be obtained directly from Eqs. 12 and 13. Graphical representation of skewness and kurtosis when $\alpha = 1.2$, $\lambda = 4.2$

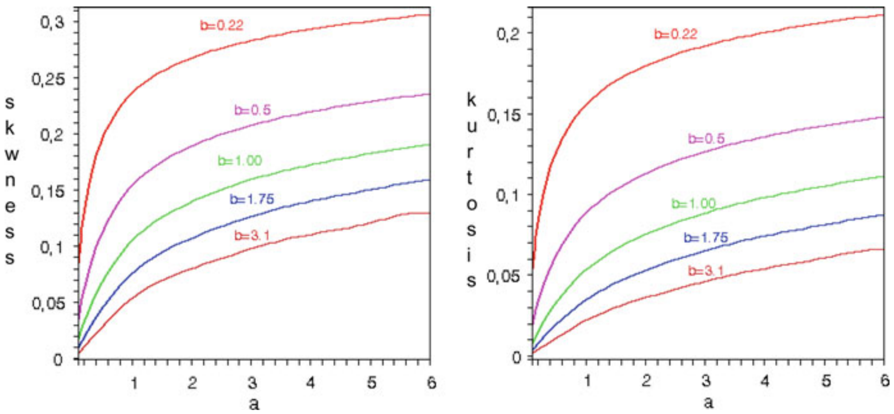

Fig. 5 Skewness and kurtosis of the BMW distribution as a function of parameter a, for some values of parameter b

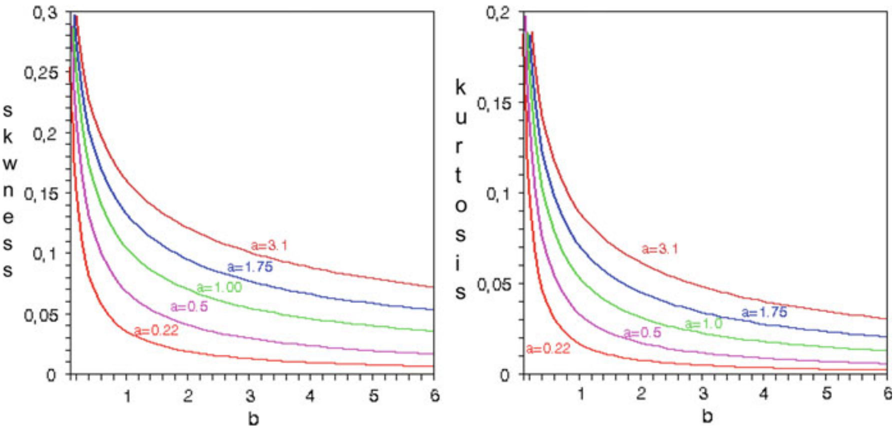

Fig. 6 Skewness and kurtosis of the BMW distribution as a function of parameter b, for some values of parameter a

and $\gamma = 3.2$, as a function of parameter a for some choices of parameter b, and as a function of parameter b for some choices of parameter a, are given in Figs. 5 and 6, respectively. These plots show that the skewness and kurtosis increase when b decreases for fixed a and when a increases for fixed b.

6 Moments of order statistics

The density of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, in a random sample of size n from the BMW distribution, is given by (for $i = 1, \ldots, n$)

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) F(x)^{i-1} \{1 - F(x)\}^{n-i}. \tag{14}$$

The cdf of the ith order statistic is simply $F_{i:n}(x) = I_{F(x)}(i, n-i+1)$. Alternatively, we can write $F_{i:n}(x)$ as binomial sums

$$F_{i:n}(x) = \sum_{k=1}^{n} \binom{n}{k} F(x)^k \{1 - F(x)\}^{n-k} = F(x)^i \sum_{k=0}^{n-i} \binom{i+k-1}{k} \{1 - F(x)\}^k .$$

We can obtain a closed form expression for the moments of the BMW order statistics using a general result due to Barakat and Abdelkader (2004) applied to the independent and identically distributed case. For a distribution with pdf $f(x)$ and cdf $F(x)$ we can write

$$E(X_{i:n}^r) = r \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} I_j(r), \tag{15}$$

where

$$I_j(r) = \int_0^\infty x^{r-1} \{1 - F(x)\}^j \mathrm{d}x.$$

Expansion (9) for the BMW survival function yields

$$1 - F(x) = \sum_{s=0}^{\infty} w_s u^{b+s},$$

where u was defined in Sect. 4. Thus,

$$I_j(r) = \int_0^\infty x^{r-1} \left(\sum_{s=0}^{\infty} w_s u^{b+s} \right)^j .$$

We now use an equation of Gradshteyn and Ryzhik (2000, Sect. 0.314) for power series raised to integer powers. For any j positive integer, we have

$$\left(\sum_{s=0}^{\infty} w_s u^s \right)^j = \sum_{s=0}^{\infty} c_{j,s} u^s, \tag{16}$$

where the coefficients $c_{j,s}$, for $s = 1, 2, \ldots,$ are easily obtained from the recurrence equation

$$c_{j,s} = (s a_0)^{-1} \sum_{m=1}^{s} (jm - s + m) w_m c_{j,s-m} \tag{17}$$

with $c_{j,0} = w_0^j$. Using the representation (17), the coefficients $c_{j,s}$ are calculated from $c_{j,0}, \ldots, c_{j,s-1}$ and, therefore, from the constants $w_0, \ldots, w_s$. The coefficients $c_{j,s}$ can be given explicitly in terms of these constants, although it is not necessary for programming numerically our expansions in any algebraic or numerical software such as Matlab, Maple or Mathematica.

From the last integral and (16) we have

$$I_j(r) = \int_0^\infty x^{r-1} \left(\sum_{s=0}^\infty w_s u^{b+s} \right)^j \mathrm{d}x = \sum_{s=0}^\infty c_{s,j} \int_0^\infty x^{r-1} u^{bj+s} \mathrm{d}x.$$

Thus,

$$I_j(r) = \sum_{s=0}^\infty c_{j,s} \int_0^\infty x^{r-1} \exp\{-\alpha(bj+s) x^\gamma \exp(\lambda x)\} \mathrm{d}x. \tag{18}$$

We obtain $I_j(r)$ using the same algebraic development by Carrasco et al. (2008, Sect. 4). We can invert the transformation $y = x^\gamma e^{\lambda x}$ to obtain x as a polynomial function in y when both λ and γ are positive. We have

$$x = \frac{\gamma}{\lambda} F\left(\frac{\lambda y^{1/\gamma}}{\gamma} \right), \tag{19}$$

where

$$F(w) = \sum_{m=1}^\infty \frac{(-1)^{m+1} m^{m-2} w^m}{(m-1)!}.$$

Hence, we can express x in terms of y from Eq. 19 as

$$x = \sum_{m=1}^\infty a_m y^{m/\gamma},$$

where

$$a_m = \frac{(-1)^{m+1} m^{m-2}}{(m-1)!} \left(\frac{\lambda}{\gamma} \right)^{m-1}. \tag{20}$$

The integral (18) written in terms of y becomes

$$J = \int_0^\infty \left\{ \sum_{m=1}^\infty a_m y^{m/\gamma} \right\}^{r-1} \exp\{-\alpha(bj+s) y\} \left\{ \sum_{p=1}^\infty \frac{a_p p}{\gamma} y^{p/\gamma - 1} \right\} \mathrm{d}y.$$

But

$$\left\{\sum_{m=1}^{\infty} a_m y^{m/\gamma}\right\}^{r-1} = \sum_{m_1,\ldots,m_{r-1}=1}^{\infty} a_{m_1} \ldots a_{m_{r-1}} y^{(m_1+\cdots+m_{r-1})/\gamma}.$$

Then, J can be rewritten as

$$J = \gamma^{-1} \sum_{m_1,\ldots,m_r=1}^{\infty} m_r A_{m_1,\ldots,m_r} \int_0^{\infty} y^{s_r/\gamma-1} \exp\{-(j+1)\alpha y\} \mathrm{d}y,$$

where the term

$$A_{m_1,\ldots,m_r} = a_{m_1} \ldots a_{m_r}$$

comes easily from the constants in (20) and

$$s_r = m_1 + \cdots + m_r.$$

Substituting $v = (j+1)\alpha y$ in the last integral, we have

$$J = \gamma^{-1} \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{m_r A_{m_1,\ldots,m_r}}{[(j+1)\alpha]^{s_r/\gamma}} \int_0^{\infty} v^{s_r/\gamma-1} \exp(-v) \mathrm{d}v,$$

which in terms of the gamma function reduces to

$$J = \gamma^{-1} \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{m_r A_{m_1,\ldots,m_r}}{[(j+1)\alpha]^{s_r/\gamma}} \Gamma(s_r/\gamma).$$

Combining (15) and (18), the rth moment of the order statistic $X_{i:n}$ can be expressed as

$$E(X_{i:n}^r) = \frac{r}{\gamma} \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \sum_{s=0}^{\infty} c_{j,s} \sum_{m_1,\ldots,m_r=1}^{\infty} \times \frac{m_r A_{m_1,\ldots,m_r}}{[(j+1)\alpha]^{s_r/\gamma}} \Gamma(s_r/\gamma). \tag{21}$$

Equation 21 can be applied to the seventeen submodels shown in Fig. 1. The moments of the order statistics are not so elegant as those moments of the BMW distribution, since they involve the calculation of the quantities $c_{j,s}$ in Eq. 17.

An alternative way to compute these moments follows by expressing the density of the ith order statistic of the BMW distribution as a mixture of MW densities. We have from (14)

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) \sum_{k=0}^{i-1} (-1)^k \{1 - F(x)\}^{n-i+k}.$$

The above sum becomes

$$\sum_{k=0}^{i-1} (-1)^k \left(\sum_{s=0}^{\infty} w_s u^{b+s} \right)^{n-i+k} = \sum_{k=0}^{i-1} (-1)^k \sum_{s=0}^{\infty} c_{n-i+k,s} u^{b(n-i+k)+s}.$$

Plugging (10) we obtain

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{j,s=0}^{\infty} \sum_{k=0}^{i-1} w_j (-1)^k c_{n-i+k,s} u^{b(n-i+k)+s} g_{\alpha(b+j),\gamma,\lambda}(x)$$

and then

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} \sum_{j,s=0}^{\infty} \sum_{k=0}^{i-1} w_j (-1)^k c_{n-i+k,s} \alpha(b+j) x^{\gamma-1} (\gamma + \lambda x) \times \exp\left\{\lambda x - \alpha \left[b(n-i+k+1) + s + j\right] x^{\gamma} e^{\lambda x}\right\}.$$

Finally, the density above can be expressed in the mixture form

$$f_{i:n}(x) = \sum_{j,s=0}^{\infty} \sum_{k=0}^{i-1} p_{n,i,k,j,s}\, g_{\alpha[b(n-i+k+1)+s+j],\gamma,\lambda}(x), \tag{22}$$

where the coefficients of the infinite linear combination are given by

$$p_{n,i,k,j,s} = p_{n,i,k,j,s}(a, b) = \frac{w_j (-1)^k (b+j) c_{n-i+k,s}}{b(n-i+k+1) + s + j}.$$

Hence, some mathematical properties (ordinary, central, inverse and factorial moments, etc.) of the BMW order statistics follow immediately from those properties of the MW distributions.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of expected order statistics. They are defined by (Hosking 1990)

$$\lambda_{r+1} = (r+1)^{-1} \sum_{k=0}^{r} (-1)^k \binom{r}{k} E(X_{r+1-k:r+1}), \quad r = 0, 1, \ldots.$$

The first four L-moments are: $\lambda_1 = E(X_{1:1})$, $\lambda_2 = \frac{1}{2} E(X_{2:2} - X_{1:2})$, $\lambda_3 = \frac{1}{3} E(X_{3:3} - 2X_{2:3} + X_{1:3})$ and $\lambda_4 = \frac{1}{4} E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$. The L-moments have the advantage that they exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. From the expansion (21) for the means ($r = 1$) of the order statistics we can obtain expansions for the L-moments of the BMW distribution.

Equations 21 and 22 are the main results of this section.

7 Maximum likelihood estimation

We now determine the maximum likelihood estimates (MLEs) of the parameters of the BMW distribution from complete samples only. Let $x_1, \ldots, x_n$ be a random sample of size n from the BMW$(a, b, \alpha, \gamma, \lambda)$ distribution. The log-likelihood function for the vector of parameters $\boldsymbol{\theta} = (a, b, \alpha, \gamma, \lambda)^T$ can be written as

$$l(\boldsymbol{\theta}) = -n \log\left[B(a, b)\right] + \sum_{i=1}^{n} \left\{\left[\log(\gamma + \lambda x_i)\right] + \log(v_i) - \log(x_i)\right\}$$
$$+(a-1) \sum_{i=1}^{n} \log\left\{1 - \exp(-v_i)\right\} - b \sum_{i=1}^{n} v_i, \tag{23}$$

where $v_i = \alpha x_i^{\gamma} \exp(\lambda x_i)$ is a transformed observation. The log-likelihood can be maximized either directly by using the SAS (PROC NLMIXED) or the program Ox (sub-routine MaxBFGS)(see, Doornik 2007) or by solving the nonlinear likelihood equations obtained by differentiating (23). The components of the score vector $U(\boldsymbol{\theta})$ are given by

$$U_a(\boldsymbol{\theta}) = -n\left[\psi(a) + \psi(a+b)\right] + \sum_{i=1}^{n} \log\left[1 - \exp(-v_i)\right],$$

$$U_b(\boldsymbol{\theta}) = -n\left[\psi(b) + \psi(a+b)\right] - \sum_{i=1}^{n} v_i,$$

$$U_\alpha(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{x_i^{\gamma} \exp(\lambda x_i)}{v_i} + (a-1) \sum_{i=1}^{n} \frac{\exp(-v_i) x_i^{\gamma} \exp(\lambda x_i)}{\left[1 - \exp(-v_i)\right]} - b \sum_{i=1}^{n} x_i^{\gamma} \exp(\lambda x_i),$$

$$U_\gamma(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left[\frac{1}{\gamma + \lambda x_i} + \log(x_i)\right] + (a-1) \sum_{i=1}^{n} \frac{\exp(-v_i) v_i \log(x_i)}{\left[1 - \exp(-v_i)\right]} - b \sum_{i=1}^{n} v_i \log(x_i),$$

$$U_\lambda(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(\frac{x_i}{\gamma + \lambda x_i} + x_i\right) + (a-1) \sum_{i=1}^{n} \frac{\exp(-v_i) v_i x_i}{\left[1 - \exp(-v_i)\right]} - b \sum_{i=1}^{n} v_i x_i,$$

where $\psi(.)$ is the digamma function.

For interval estimation and hypothesis tests on the model parameters, we require the observed information matrix. The 5×5 unit observed information matrix $J = J(\boldsymbol{\theta})$ is given in the Appendix.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \quad \text{is} \quad N_5(0, I(\boldsymbol{\theta})^{-1}),$$

where $I(\boldsymbol{\theta})$ is the expected information matrix. This asymptotic behavior is valid if $I(\boldsymbol{\theta})$ is replaced by $J(\widehat{\boldsymbol{\theta}})$, i.e., the observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$. The asymptotic multivariate normal $N_5(0, J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence intervals and confidence regions for the individual parameters and for the hazard and survival functions. The asymptotic normality is also useful for testing goodness of fit of the BMW distribution and for comparing this distribution with some of its special submodels using one of the three well-known asymptotically equivalent test statistics: the likelihood ratio (LR) statistic, Wald and Rao score statistics.

We can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the LR statistics for testing some submodels of the BMW distribution. For example, we may use the LR statistic to check if the fit using the BMW distribution is statistically "superior" to a fit using the GMW, MW and EW distributions for a given data set. In any case, hypothesis testing of the type $H_0 : \psi = \psi_0$ versus $H : \psi \neq \psi_0$, where ψ is a vector formed with some components of θ and ψ_0 is a specified vector, can be performed using any of the above three asymptotically equivalent statistics. For example, the test of $H_0 : a = b = 1$ versus $H : H_0$ *is not true* is equivalent to compare the BMW distribution with the MW distribution and the LR statistic reduces to

$$w = 2\{\ell(\widehat{a}, \widehat{b}, \widehat{\alpha}, \widehat{\gamma}, \widehat{\lambda}) - \ell(1, 1, \widetilde{\alpha}, \widetilde{\gamma}, \widetilde{\lambda})\},$$

where $\widehat{a}, \widehat{b}, \widehat{\alpha}, \widehat{\gamma}$ and $\widehat{\lambda}$ are the MLEs under H and $\widetilde{\alpha}, \widetilde{\gamma}$ and $\widetilde{\lambda}$ are the estimates under H_0.

8 Application

Here, we illustrate the superiority of the new distribution as compared to some of its submodels and also to the alternative GF and GG distributions. We consider the widely used data from Aarset (1987), and also reported in Mudholkar and Srivastava (1993); Mudholkar et al. (1996) and Wang (2000), on lifetimes of 50 components, which possess a bathtub-shaped failure rate property. The required numerical evaluations are implemented using the Ox sub-routine MaxBFGS and SAS (PROC NLMIXED). The empirical scaled TTT transform (Aarset 1987) can be used to identify the shape of the hazard function. The TTT plot for the Aarset's data in Fig. 7 shows abathtub-shaped hazard rate function and, therefore, indicates the appropriateness of the BMW distribution to fit these data.

We then perform the goodness of fit of the BMW distribution and five distributions as submodels, which allow their evaluation relative to each other and to the more

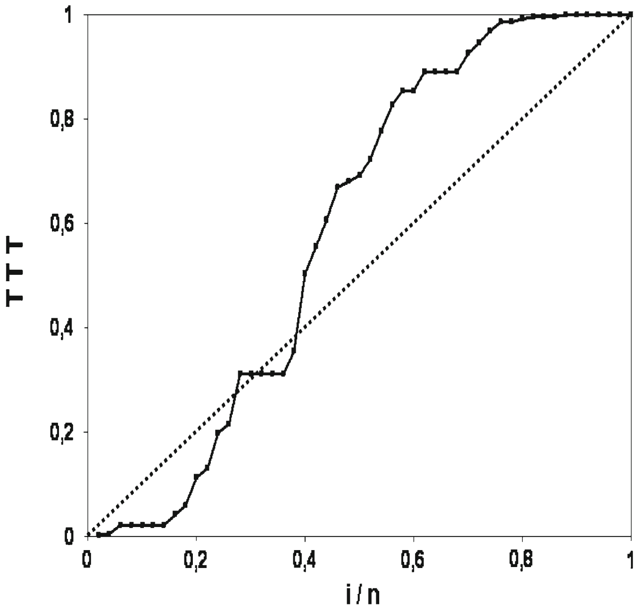

Fig. 7 TTT plot for the Aarset data

general BMW model. Further, we calculate the maximum values of the unrestricted and restricted log-likelihoods to obtain the LR statistics for testing some submodels. An analysis under the BMW model provides a check on the appropriateness of the GMW, MW and EW models and indicates the extent to which inferences depend upon the model. For example, the LR statistic was obtained for testing the hypotheses H_0: $= b = 1$ versus H_1 : H_0 *is not true*, i.e. to compare the MW model with the BMW model. The LR statistic $w = 2\{-220.817 - (-227.982)\} = 14.3293$ (p value < 0.01) yields favorable indications for the BMW model.

The MLEs of the parameters (the standard errors are given in parentheses) and the values of the Akaike information criterion (AIC) and Bayesian Information Criterion (BIC) for the six fitted models are calculated in Table 1. As we can see from these numerical results, the AIC and BIC of the BMW model are the smallest among those of the six fitted models, and hence our new model can be chosen as the best model.

Additionally, the corresponding survival function for the six fitted distributions and the well-developed Kaplan–Meier product limit estimate are plotted in Fig. 8. It can be seen that the BMW distribution is a very competitive model for describing the bathtub-shaped failure rate of the Aarset data. Further, the plots of the estimated densities and the histogram of the Aarset data given in Fig. 9 show that the BMW distribution produces a better fit than the other five submodels.

An alternative approach for modeling these data can be based on the GF distribution that exhibits a bathtub-shaped hazard rate function for the limiting case of the GG distribution. There are various parametrizations for the GF family but we consider the one proposed by Cox (2008). The density function in this parametrization can be expressed in terms of the shape parameters $p \geq 0$ and $q \in \Re$, location parameter $\beta \in \Re$ and scale parameter $\sigma > 0$ as

Table 1 Estimates of the parameters for some models fitted to the Aarset data (the standard errors are given in parentheses) and the values of the AIC and BIC statistics

Model	MLEs					AIC	BIC
	a	b	α	λ	γ		
BMW	0.1975	0.1647	0.0002	0.0541	1.3771	451.6	461.2
	(0.0462)	(0.0830)	(6.6931e−005)	(0.0157)	(0.3387)		
BW	0.18356	0.0748	0.0007	0	2.3615	463.9	471.6
	(0.0509)	(0.0353)	(0.0004)	0	(0.1715)		
GMW	0.2975	1	0.0002	0.0529	0.9942	455.8	463.4
	(0.0613)		(0.0001)	(0.0138)	(0.2396)		
MW	1	1	0.0624	0.0233	0.3548	460.3	466.0
			(0.0267)	(0.0048)	(0.1127)		
EW	0.4668	1	0.0011	0	1.5936	480.5	486.2
	(0.0889)		(0.0010)		(0.1858)		
GR	0.3643	1	0.0002	0	2	475.9	479.7
	(0.0624)		(4.8738e−005)				

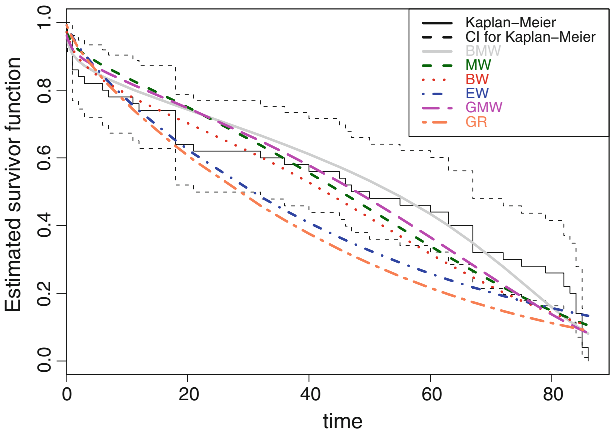

Fig. 8 Estimated survival functions for six fitted models and the empirical survival function for Aarset data

$$f(x) = \frac{\delta \exp(\frac{-\beta m_1 \delta}{\sigma}) x^{(\frac{\delta m_1}{\sigma})} (\frac{m_1}{m_2})^{m_1}}{x \sigma B(m_1, m_2)[1 + (m1/m2)\{\exp(-\beta)\, x\}^{\delta/\sigma}]^{(m_1+m_2)}}, \tag{24}$$

where $m_1 = 2[q^2 + 2p + q(q^2 + 2p)^{1/2}]^{-1}$, $m_2 = 2[q^2 + 2p - q(q^2 + 2p)^{1/2}]^{-1}$ and $\delta = (m_1^{-1} + m_2^{-1})^{1/2}$. We have the inverse relations $p = 2/(m_1 + m_2)$ and $q = \left(\frac{1}{m_1} - \frac{1}{m_2}\right)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1/2}$. The GG distribution is related to the GF distribution in an interesting way, i.e. it is obtained from (24) as an asymptotic case when $p = 0$. Other limiting distributions of the density above are the generalized log-logistic, log logistic, log normal, Burr types III and XII and Weibull distributions. The use of the

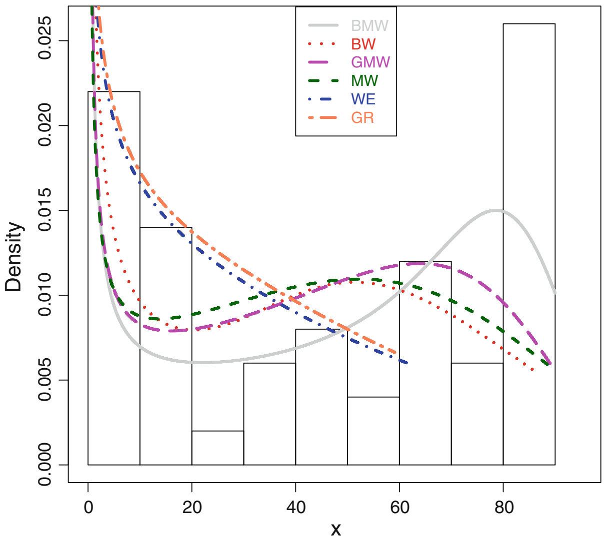

Fig. 9 Estimated densities of six models fitted to Aarset data

Table 2 Estimates of the parameters of the GF and GG distributions fitted to the Aarset data (the standard errors are given in parentheses) and the values of the AIC and BIC statistics

Model	MLEs p	β	σ	q	AIC	BIC
GF	1.0621	4.4539	0.0138	99.9255	447.8	455.5
	(89.4208)	(0.0034)	(0.0428)	(310.24)		
GG	–	4.4539	0.0138	99.9745	445.8	451.6
	–	(0.0030)	(0.0357)	(258.90)		

GF distribution provides a way to assess the fit of the GG distribution, although it has a disadvantage that, except for this special limiting distribution, its hazard function can only be decreasing or arc-shaped.

Table 2 presents the MLEs of the parameters (the standard errors are given in parentheses) for the GF and GG distributions fitted to Aarset data and the values of the statistics AIC and BIC. These numerical results are obtained using the SAS (PROC NLMIXED). The estimate of p is quite close to one but its standard error is too big, although the other estimates are nearly identical for both models. Based on the plots of the estimated GF and GG densities and the histogram of the Aarset data given in Fig. 10, we can conclude surely that both models have the same curve but do not provide a satisfactory fit.

Despite the fact that the GG model has the lowest AIC and BIC values among all fitted models, it is evident by comparing Figs. 9 and 10 that the BMW distribution could be chosen as the best model because it does fit better to the histogram of the data than the GG distribution. In the light of the above remarks, we recommend using the BMW model to fit lifetime data since it is at least efficient compared to other models available in the literature.

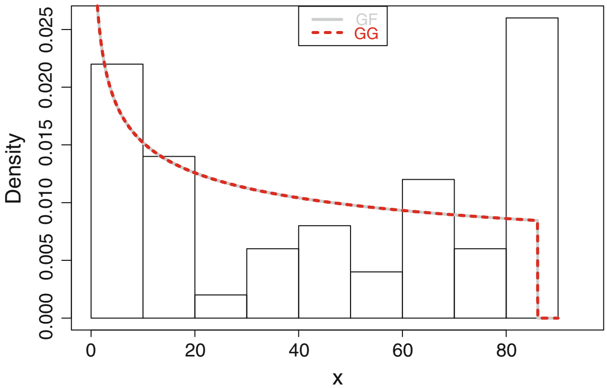

Fig. 10 Estimated densities of the GF and GG models fitted to Aarset data

9 Concluding remarks

We present a five parameter lifetime distribution, refereed to as the beta modified Weibull (BMW) distribution, which includes as special cases most of the commonly used distributions in the lifetime literature. Equally important, the new distribution is versatile and analytic tractability and accommodates all four types of hazard functions. Further, it permits testing the goodness of fit of several widely-known distributions as submodels, which is not possible in most of the generalized models used in the literature. The new model is much more flexible than the exponentiated Weibull (EW), modified Weibull (MW) and generalized modified Weibull (GMW) submodels proposed recently. The new distribution is capable of improving data fitting substantially over well-known traditional models. For this new model, powerful parametric methods such as maximum likelihood estimation and likelihood ratio tests can be effectively applied to the analysis of survival data sets to which previous attempts were unsatisfactory and unreliable due to lack of wider parametric models. We provide a mathematical treatment of the distribution including the density of the order statistics and give infinite expansions for the rth moment which hold in generality for any parameter values. We also present infinite weighted sums for the moments of the order statistics. Its flexibility, practical relevance and applicability are demonstrated using the well-known Aarset (1987) data.

Acknowledgment We are very grateful to a referee and an associate editor for helpful comments that considerably improved the paper. We gratefully acknowledge financial support from CAPES and CNPq.

Appendix

The elements of the observed information matrix $J(\boldsymbol{\theta})$ for the parameters $(a, b, \alpha, \gamma, \lambda)$ are

$$J_{aa}(\boldsymbol{\theta}) = -n\left\{\frac{\Gamma''(a)}{\Gamma(a)} - \psi^2(a) - \Gamma(a+b)\left[\psi'(a+b) + \psi^2(a+b)\right]^2 + \psi^2(a+b)\right\},$$

$$J_{bb}(\boldsymbol{\theta}) = -n\left\{\frac{\Gamma''(b)}{\Gamma(b)} - \psi^2(b) - \Gamma(a+b)\left[\psi'(a+b) + \psi^2(a+b)\right]^2 + \psi^2(a+b)\right\},$$

$$J_{\alpha\alpha}(\boldsymbol{\theta}) = -\sum_{i=1}^{n}\left[\frac{x_i^{2\gamma}\exp(2\lambda x_i)}{v_i^2}\right] - (a-1)\sum_{i=1}^{n}\left\{\frac{\exp(-v_i)x_i^{2\gamma}\exp(2\lambda x_i)}{\left[1-\exp(-v_i)\right]^2}\right\},$$

$$J_{\gamma\gamma}(\boldsymbol{\theta}) = -\sum_{i=1}^{n}(\gamma + \lambda x_i)^{-2} + (a-1)\sum_{i=1}^{n} y_i\left[\log(x_i)\right]^2 - b\sum_{i=1}^{n} v_i\left[\log(x_i)\right]^2,$$

$$J_{\lambda\lambda}(\boldsymbol{\theta}) = -\sum_{i=1}^{n}\left(\frac{x_i}{\gamma + \lambda x_i}\right)^2 + (a-1)\sum_{i=1}^{n} y_i x_i^2 - b\sum_{i=1}^{n} v_i x_i^2,$$

$$J_{\alpha\gamma}(\boldsymbol{\theta}) = (a-1)\sum_{i=1}^{n} y_i v_i^{-1} x_i^{\gamma}\exp(\lambda x_i)\log(x_i) - b\sum_{i=1}^{n} x_i^{\gamma}\exp(\lambda x_i)\log(x_i),$$

$$J_{\alpha\lambda}(\boldsymbol{\theta}) = (a-1)\sum_{i=1}^{n} y_i v_i^{-1}\exp(\lambda x_i)x_i^{\gamma+1} - b\sum_{i=1}^{n} x_i^{\gamma+1}\exp(\lambda x_i),$$

$$J_{\gamma\lambda}(\boldsymbol{\theta}) = (a-1)\sum_{i=1}^{n} y_i x_i\log(x_i) - b\sum_{i=1}^{n} v_i x_i\log(x_i),$$

$$J_{a\lambda}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{\exp(-v_i)v_i x_i}{\left[1-\exp(-v_i)\right]},$$

$$J_{a\alpha}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{\exp(-v_i)x_i^{\gamma}\exp(\lambda x_i)}{\left[1-\exp(-v_i)\right]},\ J_{a\gamma}(\boldsymbol{\theta}) = \sum_{i=1}^{n}\frac{\exp(-v_i)v_i\log(x_i)}{\left[1-\exp(-v_i)\right]},$$

$$J_{ab}(\boldsymbol{\theta}) = -n\psi'(a+b),$$

$$J_{b\alpha}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} x_i^{\gamma}\exp(\lambda x_i),\quad J_{b\gamma}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} v_i\log(x_i),\quad J_{b\lambda}(\boldsymbol{\theta}) = -\sum_{i=1}^{n} v_i x_i,$$

where $y_i = \frac{\exp(-v_i)v_i}{1-\exp(-v_i)}\left[1 - \frac{v_i}{1-\exp(v_i)}\right]$ is another transformed observation, Γ”(.) is the second-derivative of the gamma function and $\psi'(.)$ is the derivative of the digamma function.

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